Black Hole Entropy and Finite Geometry.

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The plan of this lecture

1. Motivation

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2. Mermin Squares

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3. Generalized quadrangles, Pauli operators, automorphism groups

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6. E_7, the tripartite entanglement of seven qubits and the Hamming code.

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3. Generalized quadrangles, Pauli operators, automorphism groups
4. Finite geometric interpretation of black hole entropy formulas
5. Truncations, U-duality groups.
6. E_7, the tripartite entanglement of seven qubits and the Hamming code.
7. Outlook

Motivation

"The discovery of duality symmetries in string theory has led to spectacular progress in our understanding of non-perturbative aspects of the theory. However, we still do not have a deep understanding of the meaning of these symmetries... ...A clear appreciation of symmetry principles is a sacred principle of physics. Given any physical system, we should formulate the theory in a way that makes all of the symmetries manifest.... ...One would like to have a formulation of string theory in which all of the duality symmetries are classically visible. ...There is a classical geometric system which shares all the U- duality symmetries of M-theory compactified on rectangular tori... ...The U-duality group of M-theory on T^k for rectangular compactifications with no C field vevs is given by the Weyl group $W(E_k)$. It is mapped to a subgroup of the global diffeomorphisms of the del Pezzo surfaces $\mathbb{B}_k$...."

(Iqbal, Neitzke and Vafa 2001, arXiv:0111068)

Motivation

Here we would like to show that the Weyl groups $W(E_6)$ and $W(E_7)$ show up naturally as automorphism groups of finite geometric structures associated to simple few qubit systems. Surprisingly, all the groups $W(E_k)$ mentioned by Iqbal et.al. can be shown to be arising naturally as automorphism groups of finite geometric structures associated to the four-qubit Pauli group. But here we merely restrict attention to the $k = 6, 7$ cases that provide a new way of looking at the well-known structures of $D = 4$ and $D = 5$ black hole entropy formulas.

Our starting point is the Kochen Specker theorem stating that in general it is impossible to ascribe to an individual quantum system a definite value for each set of observables not all of which necessarily commute. The point of the KS theorem is to **extract this directly from the formalism of Quantum Theory**, rather than merely appealing to percepts enunciated by the founder fathers. If such an assignment of values turned out to be possible in spite of those precepts then for instance the uncertainty relations could be viewed as manifestations of statistical scatter of some hidden variables.

Mermin Squares

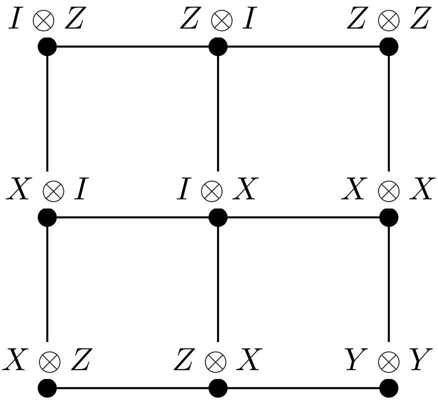
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Notation: A, B, C will be called multiqubit observables, e.g. for three qubit an example is

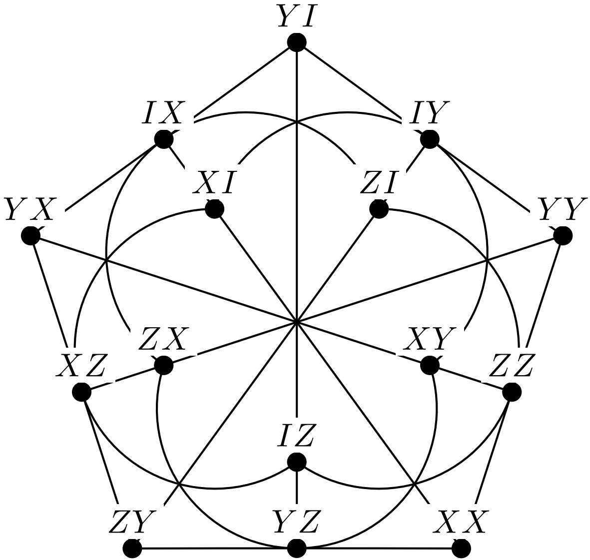
$$A = XYZ \leftrightarrow X \otimes Y \otimes Z$$

Suppose we would like to ascribe to $A, B, C, \dots$ the values $a, b, c, \dots$ to be revealed by measurements. Since any **commuting** subset of the full set of observables can be measured simultaneously, if the values are to agree with the predictions of QT they must be constrained by the condition that any relation $f(A, B, C, \dots) = 0$ holding as an identity in a **commuting subset** must also hold for their values $f(a, b, c, \dots)$. (Implicitly it is assumed that e.g. the value "a" is independent of the **context** in which A is measured.)

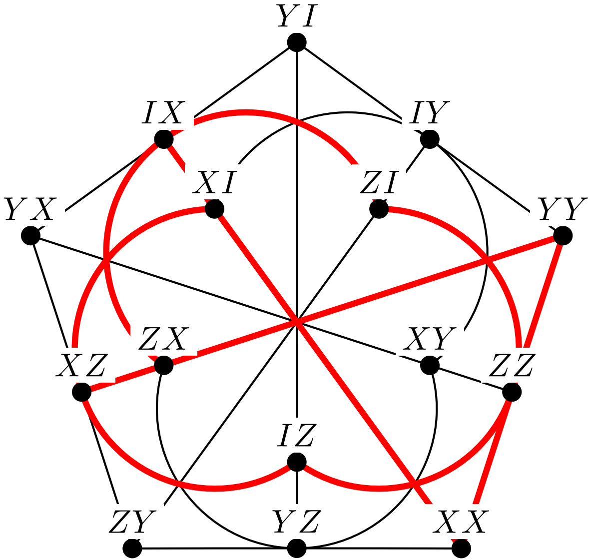
A Mermin square



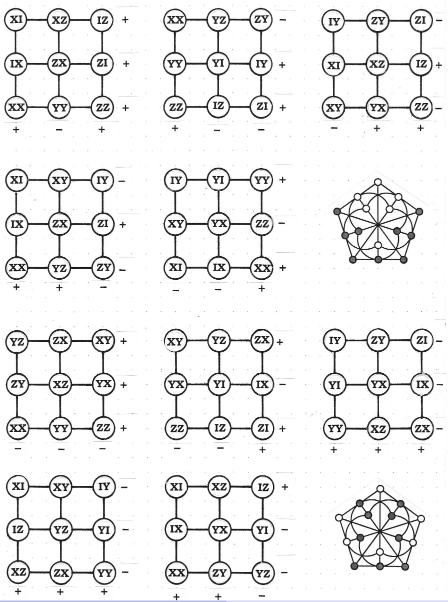
The Doily



The Doily with the Mermin square inside



The full set of Mermin squares living in the Doily



Finite generalized quadrangles $GQ(s, t)$

A *finite generalized quadrangle* of order (s, t), is an incidence structure $S = (P, B, \mathrm{I})$, where P and B are disjoint (non-empty) sets of objects, called respectively points and lines, and where I is a symmetric point-line incidence relation satisfying the following axioms:

1. each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line

It readily follows that $|P| = (s + 1)(st + 1)$ and $|B| = (t + 1)(st + 1)$.
In what follows, we shall be uniquely concerned with generalized quadrangles having lines of size *three*, GQ$(2, t)$. From a theorem of Feit and Higman it follows that we have the unique possibilities $t = 1, 2, 4$.

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1. each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line
2. each line is incident with $1 + s$ points $(s \geq 1)$ and two distinct lines are incident with at most one point

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1. each point is incident with $1 + t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line
2. each line is incident with $1 + s$ points $(s \geq 1)$ and two distinct lines are incident with at most one point
3. if x is a point and L is a line not incident with x, then there exists a unique pair $(y, M) \in P \times B$ for which $x\mathrm{I}M\mathrm{I}y\mathrm{I}L$

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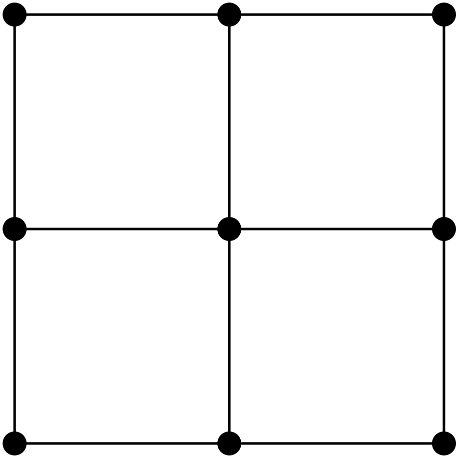
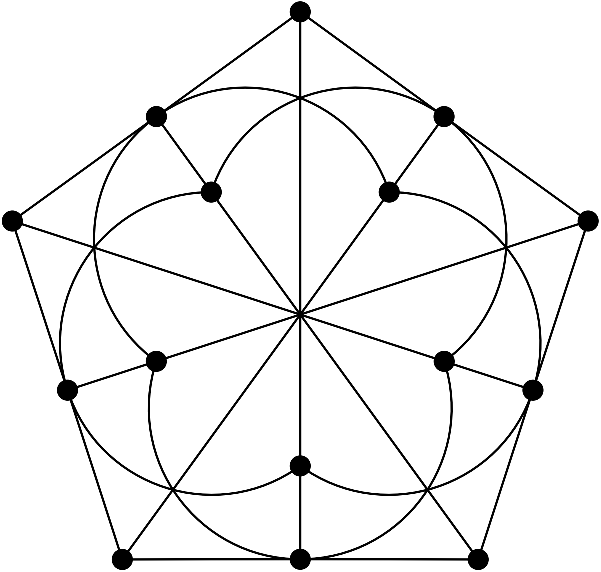
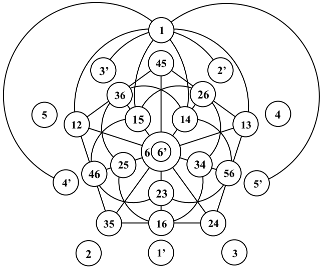
A Grid, $GQ(2,1)$

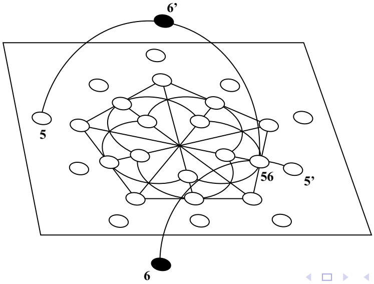


The Doily, $GQ(2,2)$



The Duad construction of $GQ(2,4)$





A **geometric hyperplane** H of a point-line incidence geometry $\Gamma(P, L)$ is a proper subset of P such that each line of Γ meets H in one or all points.
Redefine

$$Y \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then the **9 symmetric real** two-qubit "Pauli operators" form geometric hyperplanes. As another example define for a 4×4 matrix the Wootters spin-flip operation as

$$\tilde{M} \equiv (Y \otimes Y)M^T(Y \otimes Y).$$

Then we can consider from the 15 nontrivial Pauli operators the Wootters **self-dual** ones for which $\tilde{M} = M$. It turns out that we have again **9** such forming a geometric hyperplane.

The real Pauli group

The n-qubit real Pauli group is the subgroup of $GL(2^n, \mathbb{R})$ consisting of n-fold tensor products of the matrices $\pm I, \pm X, \pm Y, \pm Z$. The central quotient of the Pauli group can be given the structure of a **symplectic vectors space** $(V_n, \langle \cdot, \cdot \rangle)$ of dimension $2n$ as follows.
Let us consider the correspondence

$$I \mapsto (00), \qquad X \mapsto (01), \qquad Y \mapsto (11), \qquad Z \mapsto (10).$$

For example, XZ is taken to the 4-component vector $(0110) \in V_2 \equiv \mathbb{Z}_2{}^4$.
For two vectors $(p, q) \in V_n \times V_n$ with components $(a_1 b_1 \ldots a_n b_n)$ and $(c_1 d_1 \ldots c_n d_n)$ we define

$$\langle p, q \rangle = a_1 d_1 + b_1 c_1 + \cdots + a_n d_n + b_n c_n \qquad (1)$$

It can be shown $\langle p, q \rangle = 0$ when the corresponding real Pauli operators commute and $\langle p, q \rangle = 1$ when they anticommute.

The automorphism group of the doily $GQ(2,2)$

Label the 15 points of the doily with the 15 "duads" $\{12\}, \{13\}, \dots \{56\}$ i.e. $1 \leq i < j \leq 6$. Then the lines are the triples like $(\{12\}, \{34\}, \{56\})$ From this duad constriction it is clear that the automorphism group of $GQ(2,2)$ is the symmetric group S_6.
Let us denote by $Sp(2n, \mathbb{Z}_2)$ the symplectic group for n-qubits as the group of $2n \times 2n$ matrices with entries taken from $\mathbb{Z}_2$ that leave invariant the symplectic form of $(V_n, \langle \cdot, \cdot \rangle)$

$$Sp(2n, \mathbb{Z}_2) \equiv \{T \in M(2n, \mathbb{Z}_2) | \langle Tx, Ty \rangle = \langle x, y \rangle\}$$

Then we have

$$\mathrm{Aut}(GQ(2,2)) = S_6 \simeq Sp(4, \mathbb{Z}_2)$$

The automorphism group of $GQ(2,4)$

Let us define on V_n a **quadratic form** $Q_0 : V_n \to \mathbb{Z}_2$ as follows

$$Q_0(p) = a_1 b_1 + \cdots + a_n b_n, \qquad p \in V_n \qquad p \leftrightarrow (a_1 b_1 \dots a_n b_n)$$

Then $Q_0(p) = 0(1)$ for the Pauli operator corresponding to p being symmetric (antisymmetric). Actually one can define a whole set of quadratic forms compatible to our symplectic form as follows

$$Q_p(x) = Q_0(x) + \langle p, x \rangle, \qquad p, x \in V_n$$

One has two classes of quadratic forms: ones with $Q_0(p) = 0$, and ones with $Q_0(p) = 1$. For these correspond two classes of subgroups of $Sp(2n, \mathbb{Z}_2)$. They are called $O_\pm(2n, \mathbb{Z}_2)$, hence for instance

$$O_-(2n, \mathbb{Z}_2) \equiv \{T \in Sp(2n, \mathbb{Z}_2) | Q_p(Tx) = Q_p(x), \quad Q_0(p) = 1\}$$

The incidence structure $\mathcal{G}_n$ of the real n-qubit Pauli group is (P, L, ε) where $P = V_n - \{0\}$ and

$$L = \{\{x, y, x + y\} | x, y \in P, x \neq y, \langle x, y \rangle = 0\}$$

Points of this incidence geometry are **pairs of real Pauli operators** differing in sign, with the identity operator and its negative removed. On every line there are three points that are represented by pairwise commuting operators any two of which has the third as its product up to sign.
Obviously we have

$$\mathrm{Aut}(\mathcal{G}_n) = Sp(2n, \mathbb{Z}_2)$$

Note that $\mathcal{G}_2 = GQ(2, 2)$, having 15 points and 15 lines.

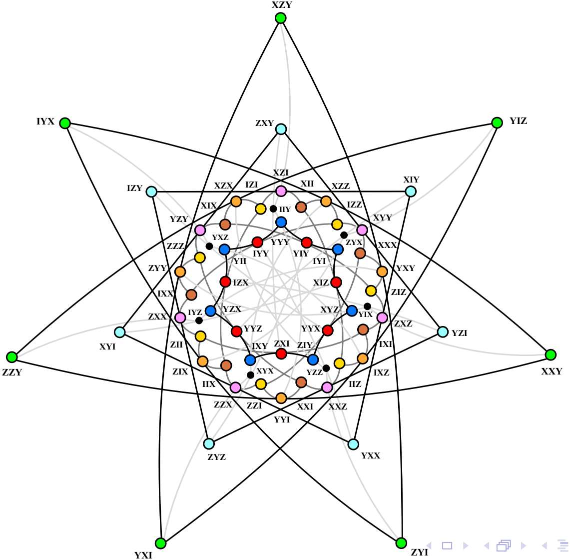
The next item in the line is $\mathcal{G}_3$ an incidence geometry which has 63 points and 315 lines. Here we meet a surprise

$$\mathrm{Aut}(\mathcal{G}_3) = Sp(6, \mathbb{Z}_2) = W(E_7)/\mathbb{Z}_2$$

In order to understand this notice that the number of generators of E_7 is $133 = 7 + 63 + 63$ hence we expect that there should exist a natural bijection between the pairs of roots and the pairs of Pauli operators differing in sign. It can be proved that it is indeed the case hence the points of $\mathcal{G}_3$ can be mapped bijectively to the pairs of roots of E_7. In this picture the Weyl reflections defined by a root α are mapped onto the so called **symplectic transvections T_p defined by an element $p \in V_3$**

$$T_p(x) = x + \langle x, p \rangle p$$

The split Cayley ($G_{2(2)}$) hexagon of order two



$GQ(2,4)$ as a geometric hyperplane of $\mathcal{G}_3$

Let us choose $IIY \leftrightarrow (000011) \in V_3 - \{0\}$ and consider the set of points

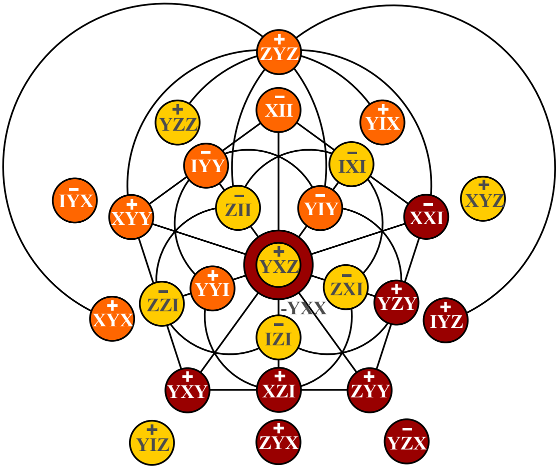
$$H_p \equiv \{x \in V_3 - \{0\} | Q_p(x) = 0\}$$

Then it can be shown that H_p is a geometric hyperplane of $\mathcal{G}_3$ having 27 points. In fact we have

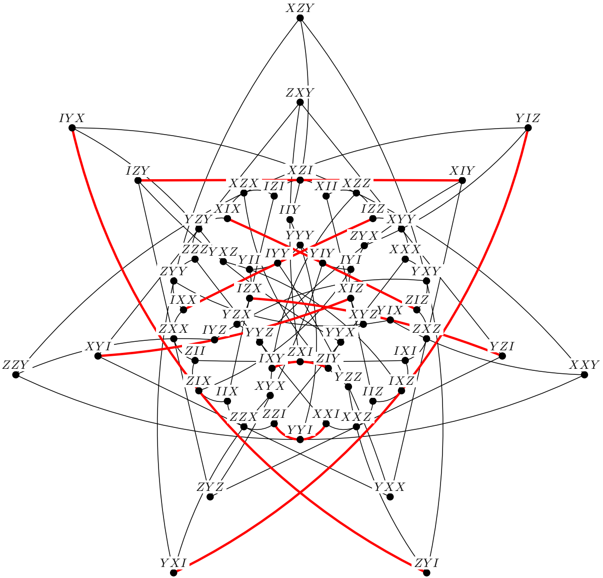
$$H_p \simeq GQ(2,4)$$

In the language of pairs of Pauli operators the above constraint yields 12 antisyimmetric operators anticommuting with IIY and 15 symmetric ones commuting with IIY. The lebelling of $GQ(2,4)$ obtained in this way can be seen on the next slide.

The labeling of $GQ(2,4)$ with three qubit Pauli operators



The hyperplane of the Hexagon with 27 points



The action of $W(E_6)$ of order 51840 on $GQ(2,4)$ is given as follows

$$O_-(6,\mathbb{Z}_2) = \langle c,d | c^2 = d^9 = (cd^2)^8 = [c,d^2]^2 = [c,d^3cd^3] = 1\rangle.$$

For the action of c

$$IXI \leftrightarrow XZI, \qquad ZYX \leftrightarrow YIX, \qquad IZI \leftrightarrow XXI$$

$$ZYZ \leftrightarrow YIZ, \qquad ZII \leftrightarrow YYI, \qquad ZYY \leftrightarrow YIY,$$

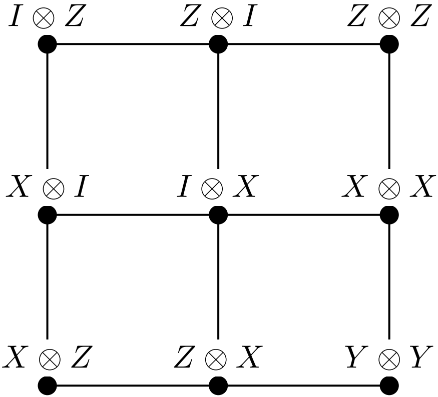
the remaining 15 operators are left invariant. For the action of d we get

$$IXI \mapsto YXZ \mapsto YZX \mapsto YIX \mapsto XYZ \mapsto IYZ \mapsto YXX \mapsto ZZI \mapsto YXY \mapsto$$

$$IZI \mapsto ZYY \mapsto XII \mapsto YZY \mapsto XYX \mapsto XYY \mapsto YIY \mapsto YIZ \mapsto IYY \mapsto$$

$$IYX \mapsto ZXI \mapsto ZYZ \mapsto ZYX \mapsto YYI \mapsto YZZ \mapsto ZII \mapsto XZI \mapsto XXI \mapsto$$

The Mermin square again



Determinant encodes the structure of Mermin

If we change $XX \mapsto -XX$ and $ZZ \mapsto -ZZ$ we have three negative signs for the rows and three positive ones for the column. This configuration is captured by the structure of the determinant of a 3×3 matrix

$$M \equiv \begin{pmatrix} \alpha & c & e \\ f & \beta & a \\ b & d & \gamma \end{pmatrix} \in M(3, \mathbb{R})$$

$$\begin{aligned} \mathcal{M} &= -\alpha X \otimes X - \beta Z \otimes Z + \gamma Y \otimes Y + aI \otimes X + bZ \otimes I + \\ &+ cZ \otimes X + dX \otimes I + eI \otimes Z + fX \otimes Z \end{aligned}$$

$$I \equiv \frac{1}{24}\mathrm{Tr}(\mathcal{M}^3) = \mathrm{Det} M$$

Determinant encodes the structure of Mermin

Clearly for $M \in M(3, \mathbb{R})$ I is an invariant under the action of the group generated by

$$M \mapsto S_1 M S_2^T, \qquad M \mapsto M^T, \qquad S_1, S_2 \in SL(3, \mathbb{R})$$

For $M \in M(3, \mathbb{Z})$ one can consider a special discrete subgroup of $SL(3, \mathbb{Z}) \times SL(3, \mathbb{Z})$ which is generated by the 4 generators of $S_3 \times S_3$ (two copies of the symmetric group), taken together with an involution exchanging the two factors. This object is the wreath product of $S_3 \wr \mathbb{Z}_2$. Now

$$\mathrm{Aut}(GQ(2,1)) = S_3 \wr \mathbb{Z}_2$$

Notice that the connection between $\mathrm{Det} M$ and $\mathrm{Tr}(\mathcal{M}^3)$ makes it possible to define a representation of $S_3 \wr \mathbb{Z}_2$ on the 9 Pauli operators. The 9 integers times these operators can be regarded as noncommutative coordinates for $GQ(2,1)$.

Black Hole Entropy in $D = 4$ and $D = 5$

The Bekenstein-Hawking entropy formula

$$S = k\frac{A}{4l_D^2}, \quad l_D^2 = \frac{\hbar G_D}{c^3}$$

for Reissner-Nordström type solutions arising from M-theory/String theory compactifications are described by **cubic** ($D = 5$) and **quartic** ($D = 4$) invariants as

$$S = \pi\sqrt{|I_3|}, \qquad S = \pi\sqrt{|I_4|}.$$

Here

$$48I_3 = \mathrm{Tr}(\Omega\mathcal{Z}\Omega\mathcal{Z}\Omega\mathcal{Z})$$

$$64I_4 = \mathrm{Tr}(\mathcal{Z}\overline{\mathcal{Z}})^2 - \frac{1}{4}(\mathrm{Tr}\mathcal{Z}\overline{\mathcal{Z}})^2 + 4(\mathrm{Pf}\mathcal{Z} + \mathrm{Pf}\overline{\mathcal{Z}}).$$

$$\mathcal{Z}_{AB} = -(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}, \quad \mathcal{Z}_{AB} = -\mathcal{Z}_{BA}, \quad A, B, I, J = 1, \ldots 8.$$

These entropy formulas containing **27** ($D = 5$) and **56** ($D = 4$) charges characterizing the black hole, are invariant under $E_{6(6)}(\mathbb{Z})$ ($D = 5$) and $E_{7(7)}(\mathbb{Z})$ ($D = 4$) respectively.

Cubic Jordan algebras and entropy formulas in $D = 5$

The charge configurations describing electric black holes and magnetic black strings of the $N = 2$, $D = 5$ ($N = 8$, $D = 5$) magic supergravities are described by cubic Jordan algebras over a division algebra $\mathbb{A}$ (or its split cousin $\mathbb{A}_s$).

$$J_3(Q) = \begin{pmatrix} q_1 & Q^v & \overline{Q^s} \\ \overline{Q^v} & q_2 & Q^c \\ Q^s & \overline{Q^c} & q_3 \end{pmatrix} \qquad q_i \in \mathbb{R}, \qquad Q^{v,s,c} \in \mathbb{A}$$

The black hole entropy is given by the cubic invariant

$$I_3(Q) = q_1 q_2 q_3 - (q_1 Q^s \overline{Q^s} + q_2 Q^c \overline{Q^c} + q_3 Q^v \overline{Q^v}) + 2\mathrm{Re}(Q^v Q^s Q^c)$$

as

$$S = \pi \sqrt{|I_3(Q)|}.$$

The groups preserving I_3 are the ones $SL(3,\mathbb{R})$, $SL(3,\mathbb{C})$, $SU^*(6)$ and $E_{6(-26)}$.
For the split octonions we have

$$Q\overline{Q}=(Q_0)^2+(Q_1)^2+(Q_2)^2+(Q_3)^2-(Q_4)^2-(Q_5)^2-(Q_6)^2-(Q_7)^2,$$

and the group preserving I_3 is $E_{6(6)}$.
The groups $E_{6(-26)}$ and $E_{6(6)}$ are the symmetry groups of the corresponding classical supergravities. In the quantum theory the black hole/string charges become integer-valued and the relevant 3×3 matrices are defined over the *integral* octonions and *integral* split octonions, respectively. Hence, the U-duality groups are in this case broken to $E_{6(-26)}(\mathbb{Z})$ and $E_{6(6)}(\mathbb{Z})$ accordingly.

Summary of patterns

Generalized quadrangles

1. $GQ(2,1)$ (grid) **9 points** and **6 lines**.

Jordan algebras (Charge configurations)

1. $J_3(\mathbb{C})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{2} = \mathbf{9}$.

Cubic invariants (Black Hole entropy)

1. $I_3(\mathbb{C})$ Number of terms: **6**. (Determinant)

Summary of patterns

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2. $GQ(2,2)$ (doily) **15 points** and **15 lines**.

Jordan algebras (Charge configurations)

1. $J_3(\mathbb{C})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{2} = \mathbf{9}$.
2. $J_3(\mathbb{H})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{4} = \mathbf{15}$.

Cubic invariants (Black Hole entropy)

1. $I_3(\mathbb{C})$ Number of terms: **6**. (Determinant)
2. $I_3(\mathbb{H})$ Number of terms: **15**. (Pfaffian)

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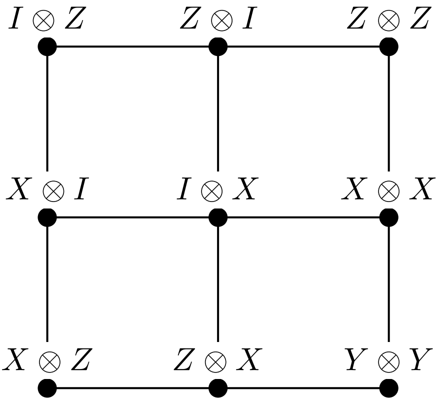
1. $GQ(2,1)$ (grid) **9 points** and **6 lines**.
2. $GQ(2,2)$ (doily) **15 points** and **15 lines**.
3. $GQ(2,4)$ **27 points** and **45 lines**.

Jordan algebras (Charge configurations)

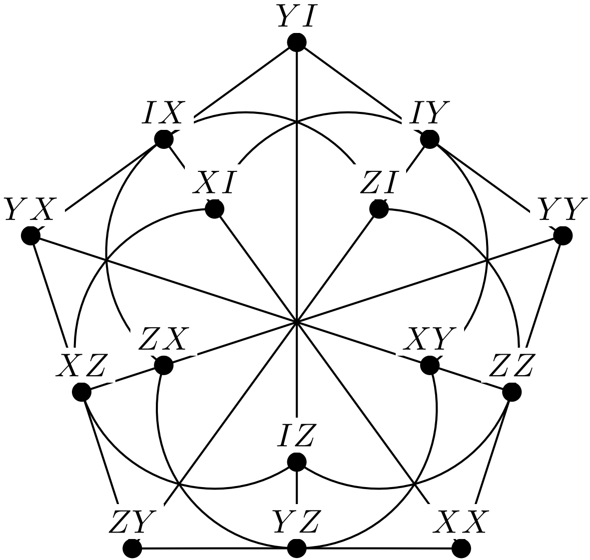
1. $J_3(\mathbb{C})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{2} = \mathbf{9}$.
2. $J_3(\mathbb{H})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{4} = \mathbf{15}$.
3. $J_3(\mathbb{O})$ Number of real numbers: $\mathbf{3} + \mathbf{3} \cdot \mathbf{8} = \mathbf{27}$.

Cubic invariants (Black Hole entropy)

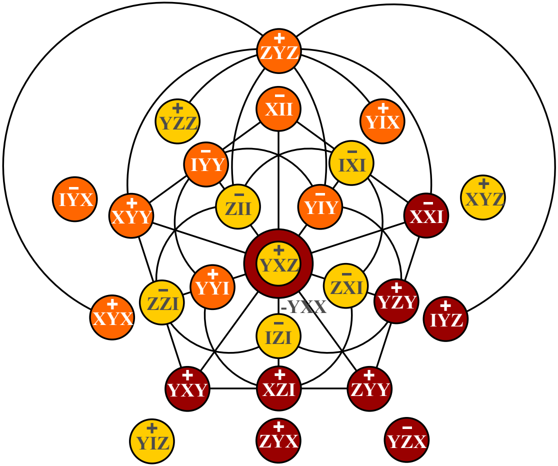
1. $I_3(\mathbb{C})$ Number of terms: **6**. (Determinant)
2. $I_3(\mathbb{H})$ Number of terms: **15**. (Pfaffian)
3. $I_3(\mathbb{O})$ Number of terms: **45**.


$I \otimes Z$
$Z \otimes I$
$Z \otimes Z$
$X \otimes I$
$I \otimes X$
$X \otimes X$
$X \otimes Z$
$Z \otimes X$
$Y \otimes Y$

The Doily, $GQ(2,2)$



The labeling of $GQ(2,4)$ with three qubit Pauli operators



The cubic invariant and the duad construction

$$E_{6(6)} \supset SL(2) \times SL(6)$$

under which

$$\mathbf{27} \to (\mathbf{2}, \mathbf{6}') \oplus (\mathbf{1}, \mathbf{15}).$$

This decomposition is displaying nicely its connection with the duad construction of GQ(2, 4). Under this decomposition I_3 factors as

$$I_3 = \mathrm{Pf}(A) + u^T A v,$$

where u and v are two six-component vectors and for the 6×6 antisymmetric matrix A we have

$$\mathrm{Pf}(A) \equiv \frac{1}{3!2^3} \varepsilon_{ijklmn} A^{ij} A^{kl} A^{mn}.$$

We also have the decomposition

$$E_{6(6)} \supset SL(3,\mathbb{R})_A \times SL(3,\mathbb{R})_B \times SL(3,\mathbb{R})_C$$

under which

$$\mathbf{27} \to (\mathbf{3}',\mathbf{3},\mathbf{1}) \otimes (\mathbf{1},\mathbf{3}',\mathbf{3}') \otimes (\mathbf{3},\mathbf{1},\mathbf{3}).$$

The above-given decomposition is related to the "bipartite entanglement of three-qutrits" interpretation of the **27** of $E_6(\mathbb{C})$. In this case we have

$$I_3 = \mathrm{Det} a + \mathrm{Det} b + \mathrm{Det} c - \mathrm{Tr}(abc),$$

where a, b, c are 3×3 matrices transforming accordingly.

Truncations

1. Truncations to 36 possible **doilies** ("quaternionic magic" with 15 charges).

Perp-sets are obtained by selecting an arbitrary point and considering all the points collinear with it. A decomposition which corresponds to perp-sets is of the form

$$E_{6(6)} \supset SO(5,5) \times SO(1,1)$$

under which

$$\mathbf{27} \to \mathbf{16_1} \oplus \mathbf{10_{-2}} \oplus \mathbf{1_4}.$$

This is the usual decomposition of the U-duality group into T duality and S duality.

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2. Truncations to 120 possible **grids** ("complex magic" with 9 charges).

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Truncations

1. Truncations to 36 possible **doilies** ("quaternionic magic" with 15 charges).
2. Truncations to 120 possible **grids** ("complex magic" with 9 charges).
3. Truncations to 27 possible **perp sets** (with 11 charges).

Perp-sets are obtained by selecting an arbitrary point and considering all the points collinear with it. A decomposition which corresponds to perp-sets is of the form

$$E_{6(6)} \supset SO(5,5) \times SO(1,1)$$

under which

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This is the usual decomposition of the U-duality group into T duality and S duality.

A $D = 4$ interpretation

Note that the decomposition

$$E_{7(7)} \supset E_{6(6)} \times SO(1,1) \qquad (2)$$

under which

$$\mathbf{56} \to \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{27}' \oplus \mathbf{1}' \qquad (3)$$

describes the relation between the $D = 4$ and $D = 5$ duality groups.

Notice that Wootters self-duality in the $N = 8$ language means that

$$\mathrm{Tr}(\Omega \mathcal{Z}) = 0, \qquad \overline{\mathcal{Z}} = \Omega \mathcal{Z} \Omega^T \quad \Omega = YYY.$$

The usual choice for $N = 8$ supergravity is $\Omega = IIY = \Gamma_1$. With this choice one can prove that

$$\Omega \mathcal{Z} = \mathcal{S} + i\mathcal{A} \equiv \frac{1}{2} x^{jk} \Gamma_{1jk} + i(y_{0j}\Gamma_{1j} - y_{1j}\Gamma_j), \qquad (4)$$

(summation for $j, k = 2, 3, \ldots, 7$).

Hence, with the notation

$$A^{jk} \equiv x^{j+1k+1}, \qquad u_j \equiv y_{0j+1}, \qquad v_j \equiv y_{1j+1}, \qquad j,k = 1,2,\dots,6,$$

we get

$$I_3 = \frac{1}{48}\mathrm{Tr}(\Omega\mathcal{Z}\Omega\mathcal{Z}\Omega\mathcal{Z}) = \mathrm{Pf}(A) + u^T A v.$$

Notice that the operators

$$\Gamma_j, \quad \Gamma_{1j}, \quad \Gamma_{1jk} \quad j,k = 2,3\dots 7$$

give rise to our noncommutative labelling, where

$$\{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4,\Gamma_5,\Gamma_6,\Gamma_7\} = \{IIY, ZYX, YIX, YZZ, XYX, IYZ, YXZ\}.$$

Hence the connection between the $D = 4$ and $D = 5$ is related to a one between the structures of GQ(2,4) and one of the geometric hyperplanes of the hexagon.

Embedding the STU model

The most general class of black holes in $\mathcal{N}=8$ supergravity/M-theory is defined by 56 charges and the entropy formula is given by the square root of the quartic Cartan-Cremmer-Julia $E_{7(7)}$ invariant

$$S = \pi\sqrt{|J_4|}$$

$$J_4 = -\mathrm{Tr}(xy)^2 + \frac{1}{4}\left(\mathrm{Tr}xy\right)^2 - 4\left(\mathrm{Pf}x + \mathrm{Pf}y\right)$$

The Cremmer-Julia form of this invariant is given in terms of the 8×8 complex central charge matrix $\mathcal{Z}$

$$J_4 = \mathrm{Tr}(\mathcal{Z}\overline{\mathcal{Z}})^2 - \frac{1}{4}(\mathrm{Tr}\mathcal{Z}\overline{\mathcal{Z}})^2 + 4(\mathrm{Pf}\mathcal{Z} + \mathrm{Pf}\overline{\mathcal{Z}})$$

$$\mathcal{Z}_{AB} = -\frac{1}{4\sqrt{2}}(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}$$

Embedding the STU model

Let us chose

$$\begin{aligned} x^{01} + iy_{01} &= -\psi_7 - i\psi_0, \qquad x^{34} + iy_{34} = \psi_1 + i\psi_6 \\ x^{26} + iy_{26} &= \psi_2 + i\psi_5, \qquad x^{57} + iy_{57} = \psi_4 + i\psi_3 \end{aligned} \tag{5}$$

Then

$$J_4 = -D(\psi) \tag{6}$$

$$Z_{canonical} = \mathrm{diag}\{z_1, z_1, z_3, z_4\} \otimes \varepsilon,$$

$$z_1 = \frac{1}{\sqrt{8}}(-\psi_7 + \psi_1 + \psi_2 + \psi_4 + i(-\psi_0 + \psi_6 + \psi_5 + \psi_3))$$

$$z_2 = \frac{1}{\sqrt{8}}(-\psi_7 - \psi_1 + \psi_2 - \psi_4 + i(-\psi_0 - \psi_6 + \psi_5 - \psi_3))$$

These results show that we should be able to obtain the three-qubit interpretation of the *STU* model as a consistent truncation of a larger entangled system living within $\mathcal{N} = 8$, $d = 4$ SUGRA.

A multiqubit representation of the 56 of E_7

A multiqubit description is possible if the complexification of $E_{7(7)}$ i.e. $E_7(\mathbb{C})$ contains the product of some number of copies of the SLOCC subgroup $SL(2,\mathbb{C})$. We indeed have

$$
\begin{aligned}
\mathbf{56} &\to (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \\
&+ (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \\
&+ (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})
\end{aligned}
$$

$$
\begin{pmatrix}
r/c & A & B & C & D & E & F & G \\
a & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
b & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
c & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
d & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
e & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
f & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
g & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\mapsto
\begin{pmatrix}
a_{ABD} \\ b_{BCE} \\ c_{CDF} \\ d_{DEG} \\ e_{EFA} \\ f_{FGB} \\ g_{GAC}
\end{pmatrix}
$$

The Fano plane and its Dual

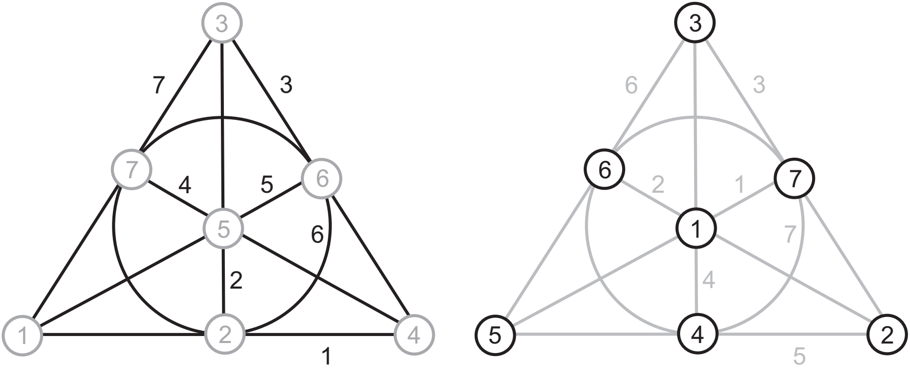


Figure: $\{1,2,3,4,5,6,7\} \leftrightarrow \{A,B,C,D,E,F,G\}$ $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7}\} \leftrightarrow \{a,b,c,d,e,f,g\}$

$$\mathcal{H} = V_{ABD} \oplus V_{BCE} \oplus V_{CDF} \oplus V_{DEG} \oplus V_{EFA} \oplus V_{FGB} \oplus V_{GAC}$$

The Cartan-Fano dictionary of Borsten et.al.

$$x^{IJ} = \begin{pmatrix} 0 & -a_7 & -b_7 & -c_7 & -d_7 & -e_7 & -f_7 & -g_7 \\ a_7 & 0 & f_1 & d_4 & -c_2 & g_2 & -b_4 & -e_1 \\ b_7 & -f_1 & 0 & g_1 & e_4 & -d_2 & a_2 & -c_4 \\ c_7 & -d_4 & -g_1 & 0 & a_1 & f_4 & -e_2 & b_2 \\ d_7 & c_2 & -e_4 & -a_1 & 0 & b_1 & g_4 & -f_2 \\ e_7 & -g_2 & d_2 & -f_4 & -b_1 & 0 & c_1 & a_4 \\ f_7 & b_4 & -a_2 & e_2 & -g_4 & -c_1 & 0 & d_1 \\ g_7 & e_1 & c_4 & -b_2 & f_2 & -a_4 & -d_1 & 0 \end{pmatrix}$$

$$y^{IJ} = \begin{pmatrix} 0 & -a_0 & -b_0 & -c_0 & -d_0 & -e_0 & -f_0 & -g_0 \\ a_0 & 0 & f_6 & d_3 & -c_5 & g_5 & -b_3 & -e_6 \\ b_0 & -f_6 & 0 & g_6 & e_3 & -d_5 & a_5 & -c_3 \\ c_0 & -d_3 & -g_6 & 0 & a_6 & f_3 & -e_5 & b_5 \\ d_0 & c_5 & -e_3 & -a_6 & 0 & b_6 & g_3 & -f_5 \\ e_0 & -g_5 & d_5 & -f_3 & -b_6 & 0 & c_6 & a_3 \\ f_0 & b_3 & -a_5 & e_5 & -g_3 & -c_6 & 0 & d_6 \\ g_0 & e_6 & c_3 & -b_5 & f_5 & -a_3 & -d_6 & 0 \end{pmatrix}.$$

There is an automorphism α of order seven which is transforming cyclically the amplitudes $a, b, \dots g$ of the relevant three qubit states into each other. α is transforming cyclically the points $1, 2, \dots 7$ of the dual Fano plane. One can find an 8×8 orthogonal matrix representation $\mathcal{D}(\alpha)$ acting on the central charge as $\mathcal{Z} \mapsto \mathcal{D}(\alpha)\mathcal{Z}\mathcal{D}^T(\alpha)$. It can be expressed in terms of the two-qubit "controlled not" (CNOT) operators

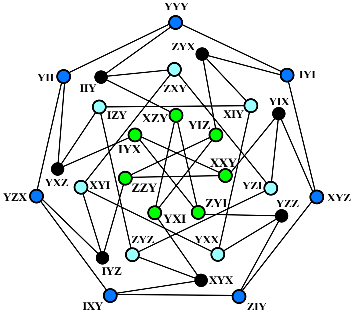
$$\mathcal{D}(\alpha) = (C_{12}C_{21})(C_{12}C_{31})C_{23}(C_{12}C_{31})$$

This representation for the automorphism of order seven can be generalized to the full automorphism group of the Fano plane.

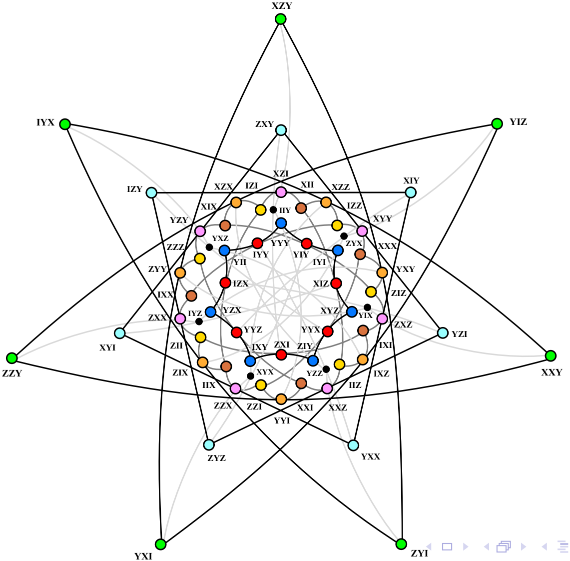
$$SL(3, \mathbb{Z}_2) \subset W(E_7) \subset E_7(\mathbb{Z})$$

$SL(3, \mathbb{Z}_2)$ can also be represented on the 28 charges regarded as composites of electric and magnetic ones with their incidence geometry corresponding to the Coxeter graph living inside $\mathcal{G}_3$. $W(E_7)$ is a subgroup of the U-duality group $E_7(\mathbb{Z})$ implementing electric-magnetic duality.

A subgeometry of the Hexagon. The Coxeter graph



The split Cayley hexagon of order two



Structure of E_7 and the Hamming code

Consider the matrix of the three-qubit discrete Fourier transformation the tensor product of three Hadamard gates where

$$H \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Delete the first column of $H \otimes H \otimes H$ and replace the -1s with 0s in the remaining 8×7 matrix.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Codeword: $(\mathbf{0}, \mathbf{1}, 0, \mathbf{1}, 0, 1, 0)$, check digits $(\mathbf{011})$, message digits (0010).

The tripartite entanglement of seven qubits and Hamming

Use the first of the two matrices of as the incidence matrix of yet another copy of the Fano plane

$$\begin{pmatrix} r/c & A & B & C & D & E & F & G \\ a & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ b & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ f & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ g & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} a_{BDF} \\ b_{ADE} \\ c_{CDG} \\ d_{ABC} \\ e_{BEG} \\ f_{AFG} \\ g_{CEF} \end{pmatrix}$$

$$(\mathcal{H}_{001}, \mathcal{H}_{010}, \dots, \mathcal{H}_{111}) \leftrightarrow (V_{BDF}, V_{ADE}, \dots, V_{CEF})$$

Reverting to reverse binary labelling

$$\mathcal{H} = V_{ABC} \oplus V_{ADE} \oplus V_{AFG} \oplus V_{BDF} \oplus V_{BEG} \oplus V_{CDG} \oplus V_{CEF}$$

Hamming labelling of Fano plane

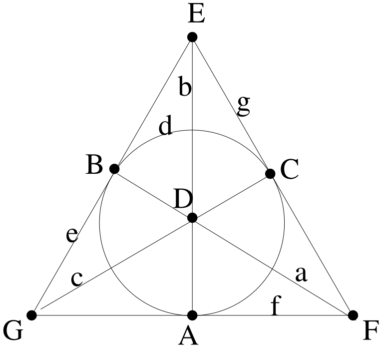


Figure: The Hamming labelling convention for the points and lines of the Fano plane.

The e_7 algebra and Hamming

We clearly have $\mathcal{W}_{000} \equiv sl(2)^{\oplus 7}$ as a subalgebra of dimension $7 \times 3 = 21$. These 21 generators are acting on $\mathcal{H}$ via the well known action of the SLOCC subgroup. The remaining 112 generators? How do they act on $\mathcal{H}$? Let us now consider the second matrix of codewords (containing **four** 1s).

$$W \equiv V_{DEFG} \oplus V_{BCFG} \oplus V_{BCDE} \oplus V_{ACEG} \oplus V_{ACDF} \oplus V_{ABEF} \oplus V_{ABDG}$$

Notice that since the complements of the quadrangles of the Fano plane are lines that can be associated to seven three-qubit states one can label each of these 16 dimensional spaces as $\mathcal{W}_{001}, \mathcal{W}_{010}, \dots \mathcal{W}_{111}$.

$$[T_{ACEG}, T_{BC'FG'}] = \Phi(ACEG, BC'FG')\varepsilon_{CC'}\varepsilon_{GG'}T_{ABEF}$$

$$e_7 = (\mathcal{W}, [\quad]), \qquad \mathcal{W} = \mathcal{W}_{000} \oplus W$$

One can see that the generators of e_7 can be written as combinations of tripartite entanglement transformations. Some of them are of SLOCC form and the others are establishing correlations between the different tripartite states . One can also show that the representation theoretic details are entirely encoded in the $(7, 3, 1)$ and $(7, 4, 2)$ designs corresponding to the two matrices of codewords which are related to lines and quadrangles of the Fano plane. The Hamming code in turn is clearly related to the Hadamard matrix which is the discrete Fourier transform on three-qubits. Such Hadamard transformations and elementary observations based on bit and phase flip errors acting on three-qubits also play a role in obtaining a nice characterization of BPS and non-BPS solutions of the STU truncation. This gives a hint that black hole solutions of more general type might be understood in a framework related to error correcting codes. **What is the connection between these observations and the recent flurry of activity on $ER = EPR$?!**

The Dual Fano Plane taking care of truncations

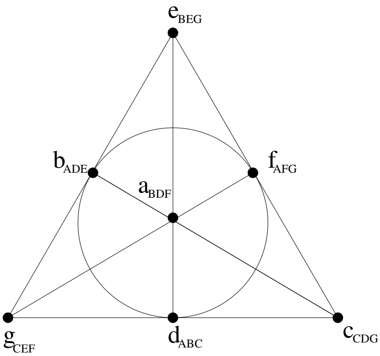


Figure: The dual Fano plane. To its points now we attached three qubit states with the representative amplitudes indicated. To the lines we associate the common qubits these tripartite states share.

The new form of Cartan's quartic invariant

$$
\begin{aligned}
J_4 = \frac{1}{2}(a^4 &+ b^4 + c^4 + d^4 + e^4 + f^4 + g^4) + \\
2[a^2 b^2 &+ b^2 c^2 + c^2 d^2 + d^2 e^2 + e^2 f^2 + f^2 g^2 + g^2 a^2 + \\
a^2 c^2 &+ b^2 d^2 + c^2 e^2 + d^2 f^2 + e^2 g^2 + f^2 a^2 + g^2 b^2 + \\
a^2 d^2 &+ b^2 e^2 + c^2 f^2 + d^2 g^2 + e^2 a^2 + f^2 b^2 + g^2 c^2] \\
+8[aceg &+ bcfg + abef + defg + acdf + bcde + abdg]
\end{aligned}
$$

$$bcde = \varepsilon^{A_1 A_3} \varepsilon^{B_3 B_4} \varepsilon^{C_2 C_3} \varepsilon^{D_1 D_2} \varepsilon^{E_1 E_4} \varepsilon^{G_2 G_4} b_{A_1 D_1 E_1} c_{C_2 D_2 G_2} d_{A_3 B_3 C_3} e_{B_4 E_4 G_4}$$

$$d^2 b^2 \equiv \mathcal{Q}(d, b), \qquad d^4 \equiv (d, d) = -2D(d)$$

$$d^2 b^2 = \varepsilon^{A_1 A_3} \varepsilon^{B_1 B_2} \varepsilon^{C_1 C_2} \varepsilon^{A_2 A_4} \varepsilon^{D_3 D_4} \varepsilon^{E_3 E_4} d_{A_1 B_1 C_1} d_{A_2 B_2 C_2} b_{A_3 D_3 E_3} b_{A_4 D_4 E_4}$$

As an example let us consider the decomposition

$$\mathbf{56} \rightarrow (\mathbf{2}, \mathbf{12}) \oplus (\mathbf{1}, \mathbf{32})$$

with respect to the maximal subgroup $SL(2, \mathbb{C}) \times SO(12, \mathbb{C})$.

$$\begin{pmatrix} d_{ABC} \\ b_{ADE} \\ f_{AFG} \end{pmatrix} \in \mathcal{H}_{(\mathbf{2},\mathbf{12})} \equiv V_{ABC} \oplus V_{ADE} \oplus V_{AFG} = V_A \otimes (V_{BC} \oplus V_{DE} \oplus V_{FG})$$

and the $(\mathbf{1}, \mathbf{32})$ part of the ones

$$\begin{pmatrix} a_{BDF} \\ e_{BEG} \\ c_{CDG} \\ g_{CEF} \end{pmatrix} \in \mathcal{H}_{(\mathbf{1},\mathbf{32})} \equiv V_{BDF} \oplus V_{BEG} \oplus V_{CDG} \oplus V_{CEF}.$$

It is also clear that by writing our representation space as

$$V_{ADE}\oplus V_A\otimes(V_{BC}\oplus V_{FG})\oplus V_D\otimes(V_{BF}\oplus V_{CG})\oplus V_E\otimes(V_{BG}\oplus V_{CF}).$$

one can easily understand the decomposition

$$(\mathbf{2},\mathbf{12})\oplus(\mathbf{1},\mathbf{32})\to(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1})\oplus(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{8_v})\oplus(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8_s})\oplus(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{8_s})$$

with respect to the inclusion
$SL(2)\times SL(2)\times SL(2)\times SO(8)\subset SL(2)\times SO(12)$.

The meaning of the $(2, 12)$ truncation in the black hole context

In this case the corresponding groups are real, hence in the supergravity approximation we have $SL(2, \mathbb{R}) \times SO(6, 6)$. We have in this case 2×12 charges. In the quantum theory they are quantized, hence they are integers and the corresponding U-duality group is $SL(2, \mathbb{Z}) \times SO(6, 6, \mathbb{Z})$. The corresponding groups are describing S and T duality transformations in toroidally compactified string theories in the low energy regime. For the black hole solutions in the corresponding models one can obtain entropy formulas that are truncations of J_4 with the amplitudes being now integers. Since we have seven lines in the dual Fano plane such truncations can be obtained in seven different ways. One can take for e.g. the line dbf in the dual Fano plane. The relevant truncation of J_4 interpreted as a measure of pure state entanglement one can take

$$J|_{(2,12)} = 2|b^4 + d^4 + f^4 + 2(b^2d^2 + d^2f^2 + b^2f^2)|$$

The meaning of the $(2,12)$ truncation in the black hole context

One can introduce a "state" describing the (NS-NS) charge configuration as

$$|\psi\rangle = \sum_{A\mu} \psi_{A\mu} |A\rangle \otimes |\mu\rangle, \quad A = 0,1, \quad \mu = 1,2,\dots 12$$

With

$$p^\mu \equiv \psi_{0\mu} = \begin{pmatrix} d_{0BC} \\ b_{0DE} \\ f_{0FG} \end{pmatrix}, \quad q^\mu \equiv \psi_{1\mu} = \begin{pmatrix} d_{1BC} \\ b_{1DE} \\ f_{1FG} \end{pmatrix}$$

$$J|_{(2,12)} = 4|(\mathbf{pp})(\mathbf{qq}) - (\mathbf{pq})^2|$$

as a measure of entanglement this relates to black hole entropy as

$$S = \frac{\pi}{2}\sqrt{J_{(2,12)}}$$

coming from the truncation of the $\mathcal{N} = 8$ case with $E_7(\mathbb{Z})$ symmetry to the $N = 4$ one of $SL(2,\mathbb{Z}) \times SO(6,6,\mathbb{Z})$.

Weyl groups as a finite subgroups of U-duality groups

The maximal supergravity in D dimensions obtained by Kaluza-Klein dimensional reduction from 11 dimension has an $E_{n(n)}(\mathbb{R})$ symmetry where $n = 11 - D$. It is conjectured that the **infinite** discrete subgroup $E_{n(n)}(\mathbb{Z})$ is an exact symmetry of the corresponding string theory, known as U-duality group. It is useful to identify a **finite** subgroup of the U-duality group that maps the fundamental quantum states of string theory among themselves. This group is $W(E_n)$ implementing electric magnetic duality. Motivated by some of the techniques of quantum information theory here we have obtained explicit realizations of $W(E_6)$ and $W(E_7)$ on finite geometries. Such structures equipped with noncommutative coordinates (Pauli operators) provide natural objects on which this physically important subgroup of the U-duality group is represented. The nice feature of this approach is that the notion of a duality-group is connected to the notion of the automorphism group of a (discretized) "space".

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