AdS black hole thermodynamics with scalar hair

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Theoretical Frontiers in Black Holes and Cosmology

Natal, 2015

Based on hep-th/0505190, hep-th/0703152 and work in progress with K. Skenderis

Motivation

- Asymptotically AdS black holes with scalar hair play a central role in applications of the AdS/CFT correspondence, especially to condensed matter physics.
- Such black holes often arise as solutions of gauged supergravities and may be uplifted to 10 or 11 dimensions.
- There has been a number of papers in the last year or two (e.g [Lü, Pang, Pope '13; Lü, Pope, Wen '14; Wen '15]) claiming that in order to satisfy the first law for certain hairy black holes one needs to introduce 'scalar charges' in the first law.
- Moreover, dyonic black holes with scalar hair have also been apparently problematic, leading to claims that in certain cases there is no well defined mass (e.g. [Chow, Compére '13]).
- Using the AdS/CFT dictionary as a guiding principle to formulate the variational problem for asymptotically AdS black holes with mixed boundary conditions on the scalars we demonstrate that such claims are not correct and the first law holds in its standard form once the conserved charges are properly defined.

Holographic dictionary = variational problem

- Constructing the holographic dictionary amounts to formulating the most general well posed variational problem for a given bulk action. This entails constructing a *symplectic space of asymptotic solutions* and determining the appropriate *boundary terms* for imposing boundary conditions compatible with the symplectic structure.
- For example, the local boundary counterterms determined through holographic renormalization are essential to have a well posed variational problem. On a conformal boundary one must formulate the boundary conditions in terms of *conformal classes* – NOT in terms of conformal representatives.
- The holographic dictionary provides a general recipe for determining *what must be kept fixed* in the variational problem (and hence in the first law for AdS black holes), as well as for obtaining the correct *boundary terms* necessary to impose the appropriate boundary conditions.

Outline

Outline

Multi-trace deformations in the large-N limit

- Consider a local, gauge-invariant, scalar single-trace operator $\mathcal{O}(x)$, transforming in some representation of the relevant rank N (gauge/global symmetry) group.
- Let us deform the theory by a generic multi-trace operator built out of $\mathcal{O}(x)$:

$$S \to S + \int d^d x f(\mathcal{O})$$

where

$$f(\mathcal{O}) = c_2 \mathcal{O}^2 + \cdots + c_k \mathcal{O}^k$$

is a polynomial in $\mathcal{O}$.

- If we insist that this is a relevant or marginal deformation of the theory, then

$$2 \leq k \leq \frac{d}{\Delta} \Rightarrow \Delta \leq \frac{d}{2}$$

- Together with unitarity this gives

$$\boxed{\frac{d}{2} - 1 \leq \Delta \leq \frac{d}{2}}$$

Normalization

- Let us normalize $\mathcal{O}$ such that $\langle\mathcal{O}\rangle = O(N^0)$ as $N \to \infty$ and assume for concreteness that $\mathcal{O}$ transforms in the adjoint representation so that $S \sim N^2$
- Generating function:

$$e^{-W[J]} = \int [d\phi] e^{-S[\phi] - N^2 \int d^d x J(x)\mathcal{O}(x)} \equiv e^{-N^2 w[J]}$$

  with

$$\sigma(x) \equiv \langle\mathcal{O}\rangle_J = \frac{\delta w[J]}{\delta J} = \mathcal{O}(N^0)$$

- Effective action:

$$e^{-\Gamma[\sigma]} = \int [dJ] e^{-N^2 w[J] + N^2 \int d^d x J(x)\sigma(x)} \equiv e^{-N^2 \bar{\Gamma}[\sigma]}$$

  with

$$J(x) = -\frac{\delta\bar{\Gamma}[\sigma]}{\delta\sigma} = \mathcal{O}(N^0)$$

- I now want to ask how do $w[J]$ and $\bar{\Gamma}[\sigma]$ transform under the multi-trace deformation

$$S[\phi] \to S_f[\phi] = S[\phi] + N^2 \int d^dx f(\mathcal{O})$$

- The answer in general depends on the details of the theory and the deformation, but large-N factorization leads to a universal result in the large-N limit...

$$
\begin{aligned}
e^{-N^2 w_f[J_f]} &= \int [d\phi] e^{-S[\phi]-N^2\int d^dx (J_f\mathcal{O}+f(\mathcal{O}))} \\
&= \int [d\phi] e^{-S[\phi]-N^2\int d^dx (J\mathcal{O}+f(\mathcal{O})-f'(\sigma)\mathcal{O})} \\
&\overset{N\to\infty}{\approx} e^{-N^2 w[J]} e^{-N^2\int d^dx (f(\sigma)-\sigma f'(\sigma))}
\end{aligned}
$$

$$
\boxed{J \equiv J_f + f'(\sigma)}
$$

$$
\begin{aligned}
e^{-N^2\bar{\Gamma}_f[\sigma]} &= \int [dJ_f] e^{-N^2 w_f[J_f] + N^2 \int d^d x J_f \sigma} \\
&\overset{N\to\infty}{\approx} \int [dJ] e^{-N^2 w[J]} e^{-N^2 \int d^d x (f(\sigma) - \sigma f'(\sigma))} e^{N^2 \int d^d x (J - f'(\sigma))\sigma} \\
&= e^{-N^2 \bar{\Gamma}[\sigma] - N^2 \int d^d x f(\sigma)}
\end{aligned}
$$

$$
[dJ_f] = [dJ]
$$

undeformed	deformed
J	$J_f = J - f'(\sigma)$
σ	$\sigma_f = \sigma$
$w[J]$	$w_f[J_f] = w[J] + \int d^dx \; (f(\sigma) - \sigma f'(\sigma))\big\|_{\sigma=\frac{\delta w}{\delta J}}$
$\bar{\Gamma}[\sigma]$	$\bar{\Gamma}_f[\sigma] = \bar{\Gamma}[\sigma] + \int d^dx f(\sigma)$

- Let us now put the theory on a general background metric $g_{(0)ij}$ so that

$$\begin{aligned} w_f[J_f] &= w[J] + \int d^dx \sqrt{-g_{(0)}} \left(f(\sigma) - \sigma f'(\sigma)\right) \\ &= w[J_f] + \int d^dx \sqrt{-g_{(0)}} f(\sigma) \end{aligned}$$

- It follows that the stress tensor

$$\langle T^{ij} \rangle = -\frac{2}{\sqrt{-g_{(0)}}} \frac{\delta w}{\delta g_{(0)ij}}$$

transforms as

$$\boxed{\langle T^{ij}\rangle[J, g_{(0)}] \to \langle T^{ij}\rangle_f[J_f, g_{(0)}] = \langle T^{ij}\rangle[J_f, g_{(0)}] - f(\sigma) {g_{(0)}}^{ij}}$$

Outline

The Dirichlet problem for AdS gravity

- Consider first the Dirichlet boundary value problem for asymptotically locally AdS solutions of the action

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}\left(R - \frac{1}{4}Z(\varphi)F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right)$$

$$V(\varphi) = -\frac{d(d-1)}{\ell^2} + \frac{1}{2}m^2\varphi^2 + \cdots$$

- Adding the Gibbons-Hawking term and the boundary counterterms, a generic variation of the total action takes the form

$$\delta(S + S_{GH} + S_{ct}) = \int d^{d+1}x(eoms)\delta\psi$$
$$+ \int d^dx\sqrt{-g_{(0)}}\left(\widehat{\pi}_{(d)}{}^{ij}\delta g_{(0)ij} + \widehat{\pi}_{(d)}{}^{i}\delta A_{(0)i} + \widehat{\pi}_{(\Delta_+)}\delta\varphi_-\right)$$

where $\widehat{\pi}_{(d)}{}^{ij}$, $\widehat{\pi}_{(d)}{}^{i}$, $\widehat{\pi}_{(\Delta_+)}$ are the renormalized radial momenta.

- Note that the counterterms are essential for the variational problem on the conformal boundary to be well posed [I.P, Skenderis '05]
- On a conformal boundary the bulk fields only induce a conformal class of boundary fields,

$$[g_{(0)}, A_{(0)}, \varphi_-]$$

with

$$(g_{(0)}, A_{(0)}, \varphi_-) \sim (e^{2\sigma(x)} g_{(0)}, A_{(0)}, e^{-(d-\Delta_+)\sigma(x)} \varphi_-)$$

and so the Dirichlet problem only makes sense in terms of conformal classes – not representatives! This is a sharp difference with the Dirichlet problem on a finite cut-off for which the Gibbons-Hawking term suffices.
- In the absence of a conformal anomaly the quantity

$$\int d^d x \sqrt{-g_{(0)}} \left(\widehat{\pi}_{(d)}{}^{ij} \delta g_{(0)ij} + \widehat{\pi}_{(d)}{}^{i} \delta A_{(0)i} + \widehat{\pi}_{(\Delta_+)} \delta \varphi_- \right)$$

is a *class function*. Rendering this boundary term (the pullback of the pre-symplectic form) a class function is the most fundamental property of the boundary counterterms. Finiteness of the on-shell action follows as a consequence.

Scheme dependence

- The local boundary counterterms are unambiguous except for the possibility of adding *finite* local boundary terms, which is interpreted as the renormalization scheme dependence in the dual theory.
- However, defining the counterterms through the variational problem imposes a severe constraint on the admissible finite local boundary terms that is not visible through the mere requirement of canceling the UV divergences:

Any finite local counterterms must be class functions, i.e. conformal invariants.

- Might have implications for supersymmetric scheme on curved boundaries [Assel, Cassani, Di Pietro, Komargodski, Lorenzen, Martelli '15]
- The scheme dependence allows the holographic charges to match with *any* other consistent way of computing the asymptotic charges by a choice of scheme, e.g. Wald Hamiltonians, Ashtekar-Magnon-Das (AMD) mass, Komar integrals. Not all of these methods give consistent conserved charges in general, however.

Ward identities

- The AdS/CFT dictionary identifies the renormalized radial canonical momenta $\widehat{\pi}_{(d)}{}^{ij}$, $\widehat{\pi}_{(d)}{}^{i}$, $\widehat{\pi}_{(\Delta_+)}$ with the one-point functions of the dual operators, namely

$$\langle T^{ij} \rangle = 2\widehat{\pi}_{(d)}{}^{ij} \quad \langle \mathcal{J}^i \rangle = -\widehat{\pi}_{(d)}{}^{i} \quad \langle \mathcal{O}_{\Delta_+} \rangle = -\widehat{\pi}_{(\Delta_+)}$$

- Using the transformation of the sources under Penrose-Brown-Henneaux bulk diffeomorphisms one finds that these one-point functions satisfy the Ward identities

$$\begin{aligned} &2D_{(0)i}\widehat{\pi}_{(d)}{}^{i}_{j} + \widehat{\pi}_{(d)}{}^{i}F_{(0)ij} - \widehat{\pi}_{(\Delta_+)}\partial_j\varphi_- = 0 \\ &D_{(0)i}\widehat{\pi}_{(d)}{}^{i} = 0 \\ &2\widehat{\pi}_{(d)}{}^{i}_{i} - (d-\Delta_+)\widehat{\pi}_{(\Delta_+)}\varphi_- = \mathcal{A}[g_{(0)}, A_{(0)}, \varphi_-] \end{aligned}$$

where $\mathcal{A}[g_{(0)}, A_{(0)}, \varphi_-]$ is the trace anomaly.

Conserved charges

- These Ward identities imply that the quantities

$$\mathcal{Q}[\xi] := \int_{\partial\mathcal{M}\cap C} \left(2\widehat{\pi}_{(d)}{}^{i}_{j} + \widehat{\pi}_{(d)}{}^{i} A_{(0)j}\right)\xi^j = \int_{\partial\mathcal{M}\cap C} \left(\langle T^i_j\rangle - \langle\mathcal{J}^i\rangle A_{(0)j}\right)\xi^j$$

are independent of the boundary Cauchy surface C iff [I.P., Skedneris '05]:

1. ξ^i is a boundary conformal Killing vector, i.e.

$$\mathcal{L}_\xi g_{(0)ij} = \frac{2}{d}(D_{(0)k}\xi^k)g_{(0)ij} \quad \mathcal{L}_\xi\varphi_- = -\frac{d-\Delta_+}{d}(D_{(0)k}\xi^k)\varphi_-, \quad \mathcal{L}_\xi A_{(0)i} = 0$$

and the trace anomaly is *numerically* zero, or

2. ξ^i is a boundary Killing vector, i.e.

$$\mathcal{L}_\xi g_{(0)ij} = 0 \quad \mathcal{L}_\xi\varphi_- = 0, \quad \mathcal{L}_\xi A_{(0)i} = 0$$

and the trace anomaly is numerically non-zero.

- These charges agree with the Wald Hamiltonians and the AMD mass, in a particular choice of scheme [I.P., Skenderis '05].

- These conserved charges satisfy the Smarr formula

$$I = \beta G(T, \Omega_i, \Phi_e) = \beta\left(M - TS - \Omega_i J_i - \Phi_e Q_e\right)$$

  and the first law

$$\delta M = T\delta S + \Omega_i \delta J_i + \Phi_e \delta Q_e$$

  for any asymptotically locally AdS black hole with Dirichlet boundary conditions on all fields [I.P., Skenderis '05].

- The only scheme-dependent observables are the free energy I and the mass M, but the scheme dependence cancels in the Smarr formula. Moreover, it does not affect the first law since the local finite counterterms are class functions.
- In the presence of a conformal anomaly

$$\delta_\sigma I = \beta \delta_\sigma M = \int d^d x \sqrt{-g_{(0)}}\mathcal{A}\delta\sigma$$

  and so the variations in the first law must ensure that the conformal representative is held fixed (important e.g. for the Kerr-AdS$_5$ black hole).

Outline

Neumann and mixed boundary conditions

- Let us parameterize the general asymptotic form of scalar field φ in asymptotically locally AdS$_{d+1}$ as (except when BF bound is saturated)

$$\varphi = \varphi_-(x)e^{-\Delta_- r} + \cdots + \varphi_+(x)e^{-\Delta_+ r} + \cdots$$

where

$$\Delta_- + \Delta_+ = d, \quad m^2\ell^2 = -\Delta_-\Delta_+$$

- For scalars with mass squared in the window

$$-\left(\frac{d}{2}\right)^2 \leq m^2 \leq -\left(\frac{d}{2}\right)^2 + 1$$

both modes φ_- and φ_+ are normalizable [Breitenlohner, Freedman '82; Balasubramanian, Kraus, Lawrence '98; Klebanov, Witten '99] and so any desirable combination of these modes can be held fixed in the variational problem.

- Note that these bounds on the mass squared are equivalent to

$$\frac{d}{2} - 1 \leq \Delta_- \leq \frac{d}{2}$$

i.e. Δ_- must satisfy the unitarity bound.

Additional boundary terms

- In order to modify the boundary conditions we start with the *renormalized* action corresponding to Dirichlet boundary conditions

$$S + S_{GH} + S_{ct}$$

and we add an extra *finite* boundary term

$$S_J[\varphi_-, \widehat{\pi}_{(\Delta_+)}]$$

such that

$$\delta\left(S + S_{GH} + S_{ct} + S_J\right) \sim \int d^d x \sqrt{-g_{(0)}} B(\varphi_-, \widehat{\pi}_{(\Delta_+)}) \delta J$$

where

$$J(\varphi_-, \widehat{\pi}_{(\Delta_+)})$$

is any function of the scalar modes that we decide to keep fixed in the variational problem.

- For Dirichlet, Neumann and Mixed boundary conditions the additional boundary term S_J respectively takes the form:

$J(\varphi_-, \widehat{\pi}_{(\Delta_+)})$	$S_J[\varphi_-, \widehat{\pi}_{(\Delta_+)}]$
$J_+ = \varphi_-$	$S_+ = 0$
$J_- = \widehat{\pi}_{(\Delta_+)}$	$S_- = -\int d^dx\sqrt{-g_{(0)}}\varphi_-\widehat{\pi}_{(\Delta_+)}$
$J_{f_-} = \widehat{\pi}_{(\Delta_+)} - f'(\varphi_-)$	$S_{f_-} = S_- - \int d^dx\sqrt{-g_{(0)}}(f(\varphi_-) - \varphi_- f'(\varphi_-))$

- Note that it is the renormalized canonical momentum

$$\widehat{\pi}_{(\Delta_+)} \sim (\Delta_+ - \Delta_-)\ell^{-1}\varphi_+ + c(\varphi_-)$$

that enters these boundary terms – not φ_+.

	Dirichlet	Neumann	Mixed
J	$J_+ \equiv \varphi_-$	$J_- \equiv \widehat{\pi}_{(\Delta_+)}$	$J_{f_-} \equiv \widehat{\pi}_{(\Delta_+)} - f'(\varphi_-)$
σ	$-\widehat{\pi}_{(\Delta_+)}$	φ_-	φ_-
$W[J]$	$-S_D[J_+]$	$-S_N[J_-]$	$-S_M[J_{f_-}]$
$\Gamma[\sigma]$	$-S_N[\widehat{\pi}_{(\Delta_+)}]$	$-S_D[\varphi_-]$	$-S_D[\varphi_-] + \int d^dx\sqrt{-g_{(0)}}f(\varphi_-)$
$\langle T^{ij}\rangle$	$2\widehat{\pi}_{(d)}{}^{ij}$	$2\widehat{\pi}_{(d)}{}^{ij} - \varphi_- J_- g_{(0)}{}^{ij}$	$2\widehat{\pi}_{(d)}{}^{ij} - \Big(f(\varphi_-) + \varphi_- J_{f_-}\Big)\, g_{(0)}{}^{ij}$

- Comparing these quantities with the large-N transformation of the generating functional, effective action and stress tensor under a multi-trace transformation one concludes that:

> *Imposing Mixed boundary conditions corresponds to deforming the theory with Neumann boundary conditions with the multi-trace deformation $f(\varphi_-)$ [Witten '01].*

- Recall that the condition for such a deformation to be relevant or marginal is

$$\frac{d}{2} - 1 \leq \Delta \leq \frac{d}{2}$$

which is only possible in the Δ_- quantization. Moreover, this condition on Δ_- coincides with the restriction on the scalar mass in order for the Δ_- quantization to be admissible.

Ward identities in the deformed theory

- Using the Ward identities of the Dirichlet problem and the above transformations leads to the following Ward identities for the deformed theory:

$$D_{(0)i}\langle T^i_j\rangle - \langle \mathcal{J}^i\rangle F_{(0)ij} + \langle \mathcal{O}_{\Delta_-}\rangle \partial_j J_{f_-} = 0$$

$$D_{(0)i}\langle \mathcal{J}^i\rangle = 0$$

$$\langle T^i_i\rangle + (d-\Delta_-)\langle \mathcal{O}_{\Delta_-}\rangle J_{f_-} = \mathcal{A}[g_{(0)}, A_{(0)}, \varphi_-] - d\left(f(\varphi_-) - \frac{\Delta_-}{d}\varphi_- f'(\varphi_-)\right)$$

- The first two Ward identities are identical in form to the Dirichlet ones, except that the sources and one-point functions have been transformed.
- Moreover, writing

$$f(\varphi_-) = \sum_k J_k \varphi_-^k$$

the trace Ward identity becomes

$$\langle T^i_i\rangle + (d-\Delta_-)J_{f_-}\langle \mathcal{O}_{\Delta_-}\rangle + \sum_k (d-\Delta_k) J_k \langle \mathcal{O}^k_{\Delta_-}\rangle = \mathcal{A}[g_{(0)}, A_{(0)}, \varphi_-]$$

where $\Delta_k = k\Delta_-$.

- The trace anomaly $\mathcal{A}[g_{(0)}, A_{(0)}, \varphi_-]$ is the same trace anomaly that appears in the Dirichlet problem. However, while in the Dirichlet problem φ_- is the source, in the Neumann and mixed problem it is the one-point function! So if the anomaly depends on φ_-, then there is an apparent problem.
- However, the fact that the scaling dimension Δ_- must satisfy the unitarity bound for Neumann or mixed boundary conditions to be admissible, together with the fact that the anomaly is a conformal invariant implies that the only way that φ_- can enter in the anomaly is

$$\mathcal{A}_{scalar} = \alpha \left(\partial^i \varphi_- \partial_i \varphi_- + \frac{(d-2)}{4(d-1)} R[g_{(0)}] \varphi_-^2 \right) + \beta \varphi_-^{\frac{d}{\Delta_-}}$$

  where $\alpha = 0$ unless $\Delta_- = (d-2)/2$ and $\beta = 0$ unless d/Δ_- is a positive integer.
- The correct prescription to obtain the generating functional and correlation functions of the operator $\mathcal{O}_{\Delta_-}$ is to use dimensional regularization, compute the final result, and only in the end set the dimension to one of the critical dimensions for which such anomalies can potentially occur [Klebanov, Witten '99]. Since there can only be scalar contributions to the anomaly at discrete values of the dimension it follows that the conformal anomaly does not contain scalar contributions.

- The Ward identities of the deformed theory suggest that conserved charges are now given by

$$\mathcal{Q}[\xi] := \int_{\partial\mathcal{M}\cap C} \left(\langle T^i_j\rangle - \langle\mathcal{J}^i\rangle A_{(0)\,j}\right)\xi^j$$

in terms of the stress tensor in the deformed theory.

- Indeed, these charges are conserved provided:
  1. ξ^i is a boundary conformal Killing vector of the deformed sources, i.e.

  $$\mathcal{L}_\xi g_{(0)\,ij} = \frac{2}{d}(D_{(0)\,k}\xi^k)g_{(0)\,ij} \quad \mathcal{L}_\xi J_{f_-} = -\frac{d-\Delta_-}{d}(D_{(0)\,k}\xi^k)J_{f_-}, \quad \mathcal{L}_\xi A_{(0)\,i} = 0$$

  the trace anomaly is *numerically* zero, and $f(\varphi_-) = \lambda\varphi_-^{d/\Delta_-}$, or

  2. ξ^i is a boundary Killing vector of the deformed sources, i.e.

  $$\mathcal{L}_\xi g_{(0)\,ij} = 0 \quad \mathcal{L}_\xi J_{f_-} = 0, \quad \mathcal{L}_\xi A_{(0)\,i} = 0$$

  and either the trace anomaly is numerically non-zero or $f(\varphi_-)$ is not marginal, or both.

Smarr formula and the first law

- These conserved charges again satisfy the Smarr formula

$$I = \beta G(T, \Omega_i, \Phi_e) = \beta\left(M - TS - \Omega_i J_i - \Phi_e Q_e\right)$$

  and the first law

$$\delta M = T\delta S + \Omega_i \delta J_i + \Phi_e \delta Q_e$$

  for any asymptotically locally AdS black hole with Dirichlet/Neumann/Mixed boundary conditions [I.P., Skenderis '15].

- All charges must be computed with the stress tensor of the deformed theory.
- The variations in the first law must keep fixed the multi-trace couplings in $f(\varphi_-)$, in addition to the single trace sources.
- AMD mass does not give the correct result for mixed boundary conditions!
- Wald Hamiltonians give the correct result provided the variations respect the boundary conditions.

Outline

- A black hole that was recently presented as an example for the necessity of scalar charges can be found in 3 dimensions ($d = 2$) for the scalar potential [Wen '15]

$$V(\varphi) = g^2(\cosh\psi)^{4\mu}\left(\mu\tanh^2\psi - 2\right) = -g^2\left(2 + \frac{3}{8}\varphi^2 + \frac{1}{32}\varphi^4 + \mathcal{O}(\varphi^5)\right)$$

where $g = 1/\ell$, $\psi = \varphi/(2\sqrt{2\mu})$, and

$$-1 < m^2\ell^2 = -3/4 < 0, \quad \Delta_- = 1/2, \quad \Delta_+ = 3/2$$

- The black hole solution takes the form

$$\varphi(r) = 2\sqrt{2\mu}\text{Arctanh}\left(\frac{1}{\sqrt{1 + r/q}}\right),$$

$$ds^2 = -g^2r^2dt^2 + \frac{dr^2}{g^2r^2(1 + q/r)^{2\mu}} + r^2d\theta^2$$

where q is the only parameter of the solution.

- The boundary counterterms are

$$S_{ct} = -\frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\left(\frac{1}{\ell}+\frac{1}{8\ell}\varphi^2\right)$$

- This leads to the renormalized metric momentum

$$\widehat{\pi}_{(2)ij} = \frac{1}{2\kappa^2}\frac{g}{6}\mu(3\mu-1)q^2\eta_{ij}$$

- Moreover, in Fefferman-Graham coordinates

$$\phi = 2\sqrt{2\mu q}e^{-g\bar{r}/2} + \frac{(3\mu-1)}{48\mu}8(2\mu q)^{3/2}e^{-3g\bar{r}/2} + \mathcal{O}(e^{-5g\bar{r}/2})$$

and so

$$\widehat{\pi}_{(\Delta_+)} = \frac{g}{2\kappa^2}(\Delta_+ - \Delta_-)\varphi_+ = f'(\varphi_-) = \frac{g}{2\kappa^2}\frac{(3\mu-1)}{48\mu}\varphi_-^3 \Rightarrow$$

$$f(\varphi_-) = \frac{g}{2\kappa^2}\frac{(3\mu-1)}{192\mu}\varphi_-^4 = \frac{g}{2\kappa^2}\frac{1}{3}\mu(3\mu-1)q^2$$

- The stress tensor vanishes identically:

$$\begin{aligned}\langle T^{ij}\rangle &= 2\widehat{\pi}_{(2)}{}^{ij} - \eta^{ij} f(\varphi_-)\\ &= 2\times\frac{1}{2\kappa^2}\frac{g}{6}\mu(3\mu-1)q^2\eta^{ij} - \frac{g}{2\kappa^2}\frac{1}{3}\mu(3\mu-1)q^2\eta^{ij} = 0\end{aligned}$$

  and so

$$M = 0$$

- This agrees with the Wald mass and is consistent with the first law

$$\delta M = T\delta S$$

  since the entropy vanishes identically too.

- Another hairy black hole in $d = 3$ (AdS_4) is the MTZ black hole [Martínez, Troncoso, Zanelli '04].
- The potential takes the form

$$V(\varphi) = -\frac{6}{\ell^2} \cosh\left(\varphi/\sqrt{3}\right)$$

and it corresponds to $m^2\ell^2 = -2, \quad \Delta_- = 1, \quad \Delta_+ = 2$

- This potential can be embedded in the $U(1)^4$ truncation of $\mathcal{N} = 8$ gauge supergravity [IP '06] and uplifted to 11-diemensional supergravity using the ansatz of [Cvetič et al. '99].

- The black hole solution takes the form

$$\varphi = \sqrt{12}\,\mathrm{arctanh}\left(\frac{G\mu}{r+G\mu}\right),$$

$$ds^2 = \frac{r(r+2G\mu)}{(r+G\mu)^2}\left(-\left(\frac{r^2}{\ell^2}-\left(1+\frac{G\mu}{r}\right)^2\right)dt^2 + \left(\frac{r^2}{\ell^2}-\left(1+\frac{G\mu}{r}\right)^2\right)^{-1}dr^2 + r^2 d\Sigma_2^2\right)$$

where Σ_2 is a compact Riemann surface of constant negative Ricci curvature

- The boundary counterterms are

$$S_{ct} = -\frac{1}{\kappa^2}\int d^dx\sqrt{-\gamma}\left(\frac{2}{\ell} + \frac{1}{4\ell}\varphi^2 + \frac{\ell}{2}R[\gamma]\right)$$

- This leads to the renormalized metric momentum

$$\widehat{\pi}_{(3)tt} = \frac{1}{2\kappa^2}\frac{2G\mu(G^2\mu^2+\ell^2)}{\ell^4},$$
$$\widehat{\pi}_{(3)\alpha\beta} = \frac{1}{2\kappa^2}\frac{G\mu(-2G^2\mu^2+\ell^2)}{\ell^4}\ell^2\sigma_{\alpha\beta}$$

- Moreover, in Fefferman-Graham coordinates

$$\varphi = \frac{2\sqrt{3}G\mu}{\ell}e^{-\bar{r}/\ell} - \frac{2\sqrt{3}G^2\mu^2}{\ell^2}e^{-2\bar{r}/\ell} + \cdots$$

and so

$$\widehat{\pi}_{(\Delta_+)} = \frac{1}{2\kappa^2\ell}(\Delta_+ - \Delta_-)\beta = f'(\alpha) = -\frac{1}{4\sqrt{3}\kappa^2\ell}\left(\frac{2\sqrt{3}G\mu}{\ell}\right)^2 \Rightarrow$$
$$f(\varphi_-) = -\frac{1}{12\sqrt{3}\kappa^2\ell}\varphi_-^3$$

- The stress tensor is given by

$$\langle T_{ij} \rangle = 2\widehat{\pi}_{(2)\,ij} - \eta_{ij} f(\varphi_-)$$

and so

$$\langle T_{tt} \rangle = \frac{2G\mu(G^2\mu^2+\ell^2)}{\kappa^2\ell^4} - \frac{1}{12\sqrt{3}\kappa^2\ell}\left(\frac{2\sqrt{3}G\mu}{\ell}\right)^3 = \frac{2G\mu}{\kappa^2\ell^2}$$

$$\langle T_{\alpha\beta} \rangle = \frac{G\mu(-2G^2\mu^2+\ell^2)}{\kappa^2\ell^4}\sigma_{\alpha\beta} + \frac{1}{12\sqrt{3}\kappa^2\ell}\left(\frac{2\sqrt{3}G\mu}{\ell}\right)^3 \sigma_{\alpha\beta} = \frac{2G\mu}{\kappa^2\ell^2}\sigma_{\alpha\beta}$$

- This gives the mass

$$M = \frac{\mu}{4\pi}\mathrm{Vol}\Sigma_2$$

in agreement with the Wald mass and the first law

$$\delta M = T\delta S$$

and the Smarr formula

$$I = \beta(M - TS)$$

Outline

- Adding a parity odd term of the form

$$S_{CS} = -\frac{1}{2\kappa^2}\int d^4x\Pi(\varphi)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

to the action simply shifts the Maxwell equation, but without any obstruction to the charge conservation equations.

- However, such a term gives rise to an anomaly in the holographic diffeomorphism Ward identity, namely [Lindgren, IP, Taliotis, Vanhoof '15]

$$D_{(0)i}\langle T^i_j\rangle - \langle\mathcal{J}^i\rangle F_{(0)ij} + \langle\mathcal{O}_{\Delta_-}\rangle\partial_j J_{f_-} = -\frac{2}{\kappa^2}\Pi(\varphi)\epsilon^{jkl}F_{ij}F_{kl}$$

- If one does not modify the definition of the spacetime charges, then one concludes that there is no mass for dyonic black holes since there is an obstruction to the conservation of the mass given by

$$-\frac{2}{\kappa^2}\Pi(\varphi)\epsilon^{jkl}F_{ij}F_{kl}\xi^i$$

where ξ^i is the timelike Killing vector. Note that this quantity is nonzero for dyonic solutions.

- However, the anomalous Ward identity allows us to generalize the definition of the holographic charges appropriately in this case as

$$\boxed{\mathcal{Q}[\xi] := \int_{\partial\mathcal{M}\cap C}\left(\langle T^i_j\rangle - \left(\langle\mathcal{J}^i\rangle + \frac{2}{\kappa^2}\Pi(\varphi)\epsilon^{ikl}F_{kl}\right)A_{(0)j}\right)\xi^j}$$

leading to a well defined conserved charge, provided

$$D_i\left(\Pi(\varphi)\epsilon^{ijk}F_{kl}\right) = 0$$

Outline

Concluding remarks

- The AdS/CFT dictionary provides a general prescription for consistently formulating the variational problem for asymptotically AdS black holes.
- The use of such general techniques is particularly important in subtle cases, such as for scalar fields satisfying mixed boundary conditions and dyonic black holes, for which other methods lead to incorrect results or apparent paradoxes.
- A proper formulation of the variational problem leads to conserved charges that are compatible with the first law in its standard form.
- No scalar charges.
- Analogous arguments apply to the boundary conditions for vector fields in AdS_4 and dyonic black holes.

Thanks!