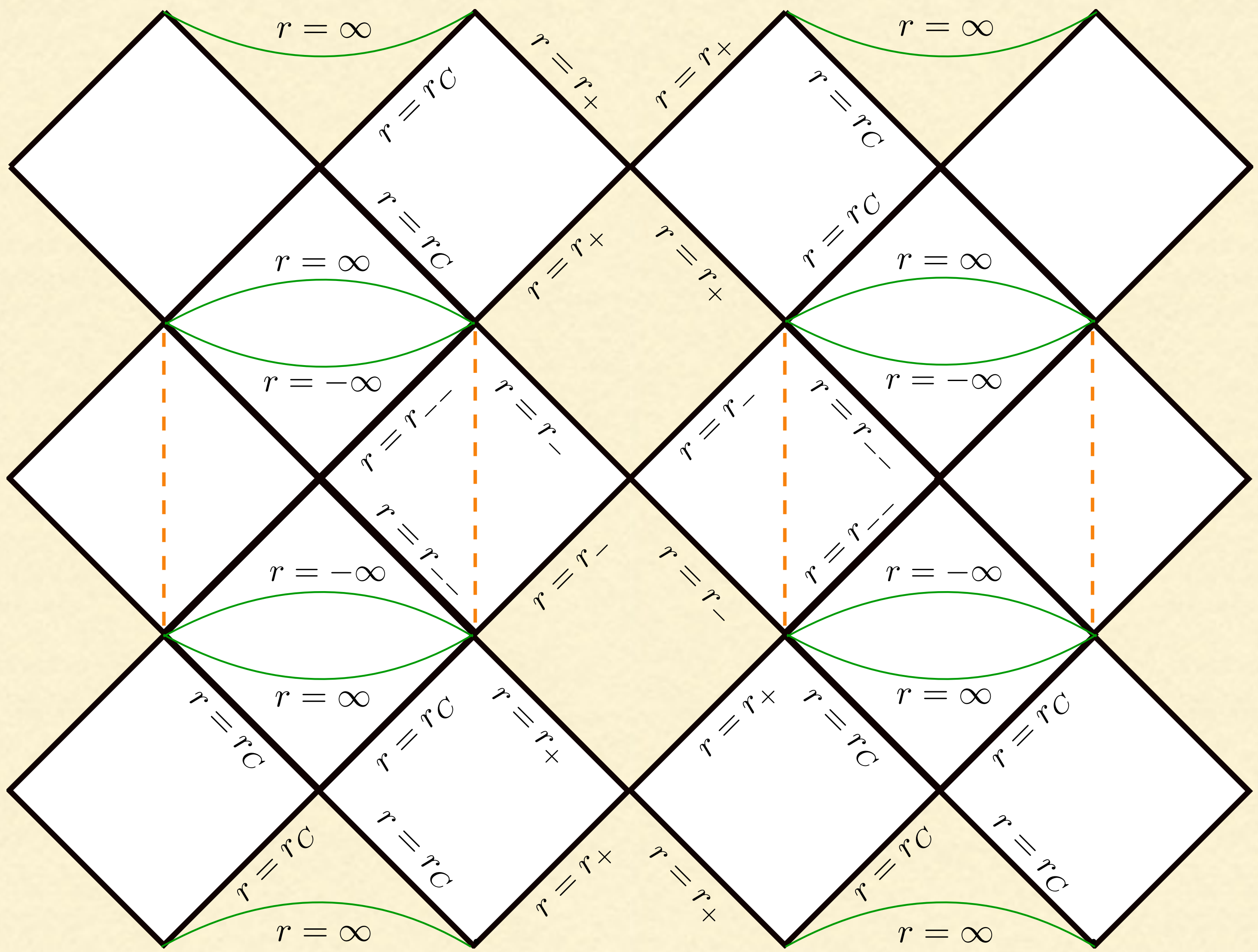


ISOMONODROMY AND BLACK HOLE SCATTERING

Bruno Carneiro da Cunha (UFPE) & Fábio Novaes (UFPE & IIP-UFRN)

Theoretical Frontiers in Black Holes and Cosmology - IIP - UFRN - 17/06/2015

AN IMPORTANT PROBLEM

- Superradiance, greybody factors
- Normal modes: adS/CFT
- Normal modes: stability of Kerr-(a)dS Solution
- Scattering between different asymptotic regions and unitarity

A DIFFICULT PROBLEM

- Sophisticated numerical methods (since the 60's!)
- New, unchartered special functions
- Unsolved problem for Teukolsky: numerics only go so far
- Non-linear stability

OUTLINE

- New method to extract scattering coefficients, based on isomonodromy
- Relations to integrable systems and Conformal Field Theory
- Exact solutions based on Painlevé transcendents.
- Near-extremal approximate solutions for Kerr-dS and Kerr-Newman black holes

$$(\nabla_a \nabla^a + \xi R)\psi = 0$$

BCC, FN: 1404.5188, 1506.XXXXX

Killing-Yano, twistor programme and separability

$$ds^2 = \sum_{\mu=1}^{n} \left[\frac{dx_\mu^2}{Q_\mu} + Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 - \frac{\epsilon c}{A^{(n)}} \left(\sum_{k=0}^{n-1} A^{(k)} d\psi_k \right)^2 \right]$$

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad A_\mu^{(j)} = \sum_{\substack{\nu_1 < \cdots < \nu_j \\ \nu_i \neq \mu}} x_{\nu_1}^2 \ldots x_{\nu_j}^2, \quad A^{(j)} = \sum_{\nu_1 < \cdots < \nu_j} x_{\nu_1}^2 \ldots x_{\nu_j}^2,$$

$$U_\mu = \prod_{\nu \neq \mu} (x_\nu^2 - x_\mu^2), \quad X_\mu = \sum_{k=\epsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\epsilon} + \frac{\epsilon c}{x_\mu^2}.$$

Many cases of interest in 4d:
Kerr, Kerr-Newman, Kerr-NUT-(a)dS, etc.

$$\psi(t,\phi,r,\theta) = e^{-i\omega t} e^{im\phi} R(r) S(\theta)$$

$$\partial_r (Q(r) \partial_r R(r)) + \left(V_r(r) + \frac{W_r^2}{Q(r)} \right) R(r) = 0 \ ,$$

$$V_r = \kappa_0 r^2 + \kappa_1, \quad W_r = \Psi_0 r^2 + \Psi_1,$$

$$\kappa_0 = -4\Lambda\xi, \quad \kappa_1 = -C_\ell,$$

$$\Psi_0 = \omega \left(1 + \frac{\Lambda a^2}{3} \right), \quad \Psi_1 = a \left(\omega \frac{(a+b)^2}{a} - m \right) \left(1 + \frac{\Lambda a^2}{3} \right)$$

A word about the angular equation

$$\partial_p(P(p)\partial_p S(p)) + \left(-4\Lambda\xi p^2 + C_\ell - \frac{(\Psi_0 p^2 - \Psi_1)^2}{P(p)}\right) S(p) = 0$$

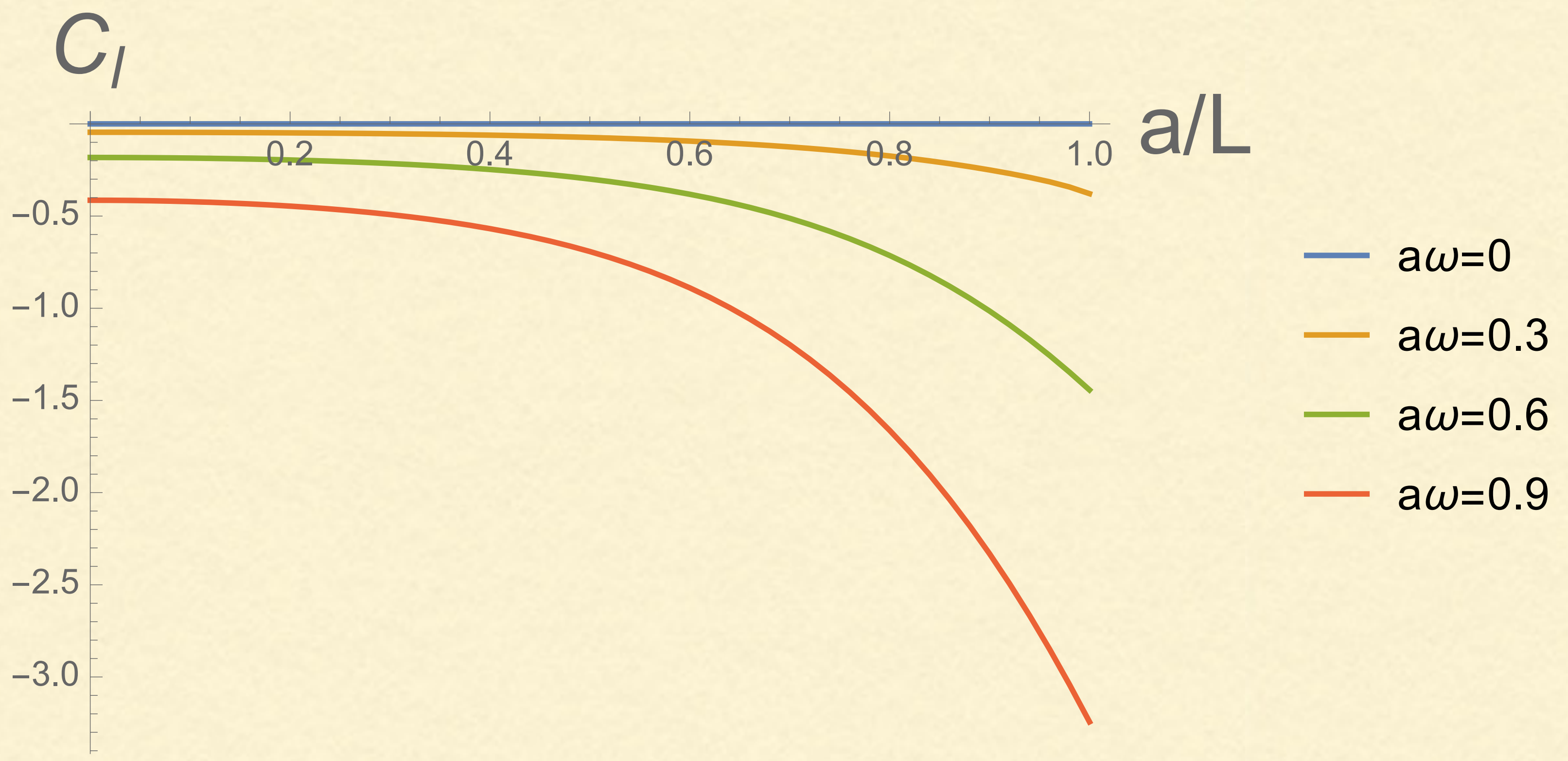
$$P(p) = -\frac{\Lambda}{3}p^4 - \epsilon p^2 + 2np + k$$

$$\epsilon = 1 - (a^2 - 6b^2)\frac{\lambda}{3}, \qquad k = (a^2 - b^2)(1 - b^2\Lambda), \qquad n = b\left[1 + (a^2 - 4b^2)\frac{\Lambda}{3}\right]$$

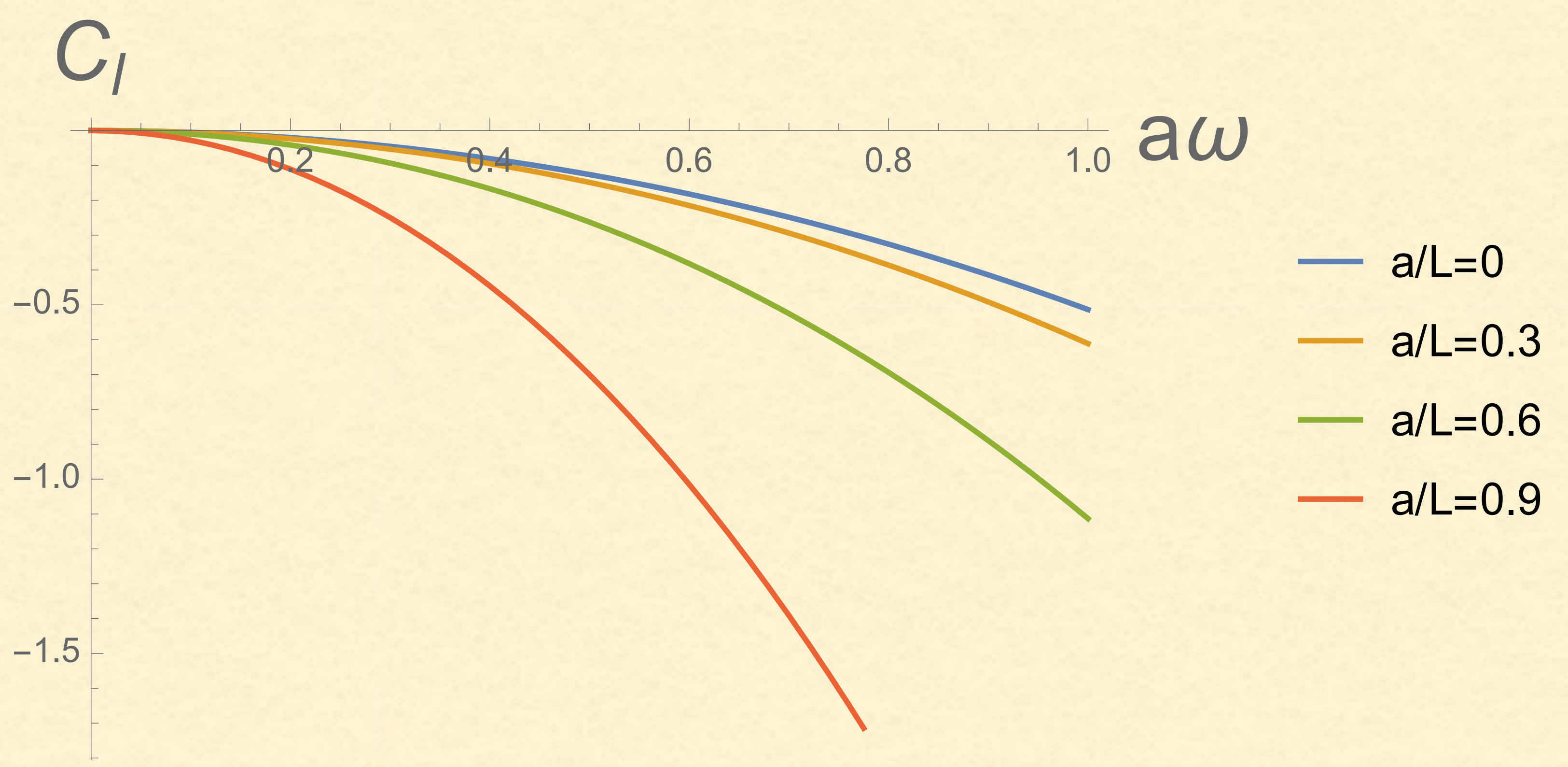
Spheroidal harmonics

Eigenvalue problem

Padé (rational) approximants



C_l
a/L
0.2
0.4
0.6
0.8
1.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
aω=0
aω=0.3
aω=0.6
aω=0.9



C_l
aω
0.2
0.4
0.6
0.8
1.0
−0.5
−1.0
−1.5
a/L=0
a/L=0.3
a/L=0.6
a/L=0.9

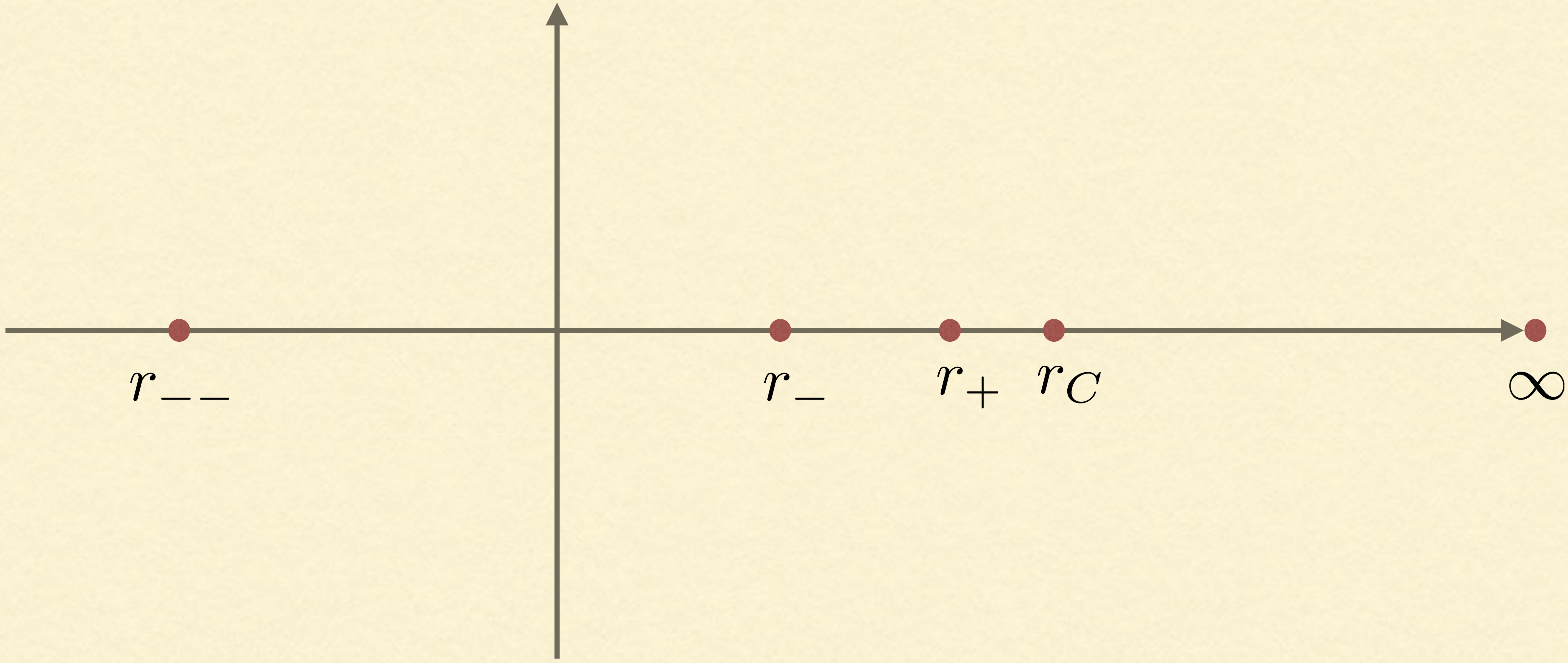
The radial equation:

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_{t_0}}{z-t_0}\right) y' + \left(\frac{q_1 q_2}{z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)}\right) y = 0$$

$$\theta_k = 2i\chi^2 \left(\frac{\omega(r_k^2+a^2)-am}{\Delta_r'(r_k)}\right) = \frac{i}{2\pi}\left(\frac{\omega-\Omega_k m}{T_k}\right), \quad k = 0, 1, t_0, \infty,$$

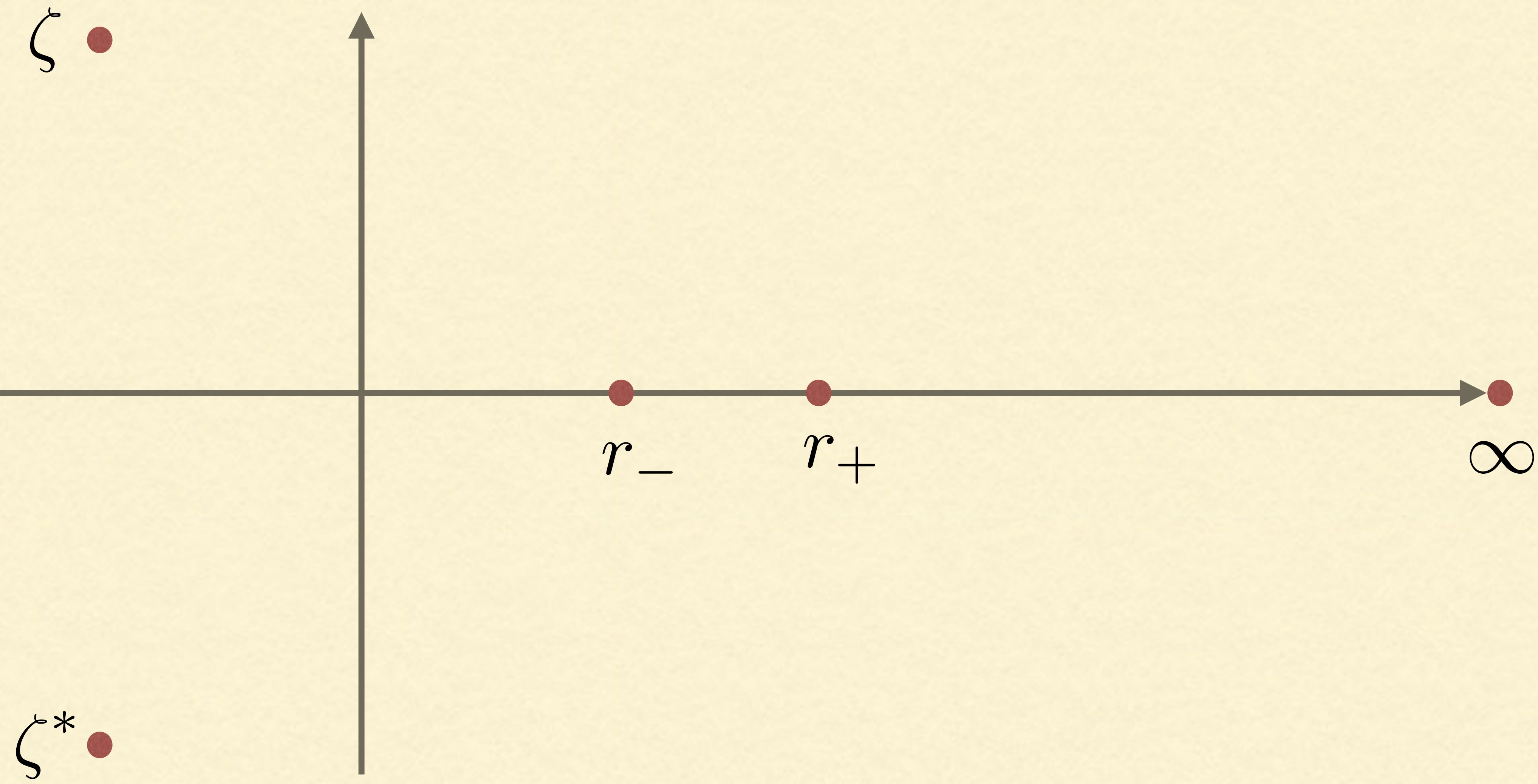
$$K_0 = -\frac{1}{t_0 - z_\infty}\left[1 + \frac{r_{t_0}-r_\infty}{\Delta_r'(r_{t_0})}\left(-\frac{2}{L^2}r_{t_0}^2 + \lambda_\ell + \chi^2(a^2\omega^2 - 2a\omega m)\right)\right.$$
$$\left. -2i\chi^2 \frac{\omega(r_{t_0}r_\infty + a^2) - am}{\Delta_r'(r_{t_0})}\right]$$

THE BRÜCKER EQUATION



de Sitter

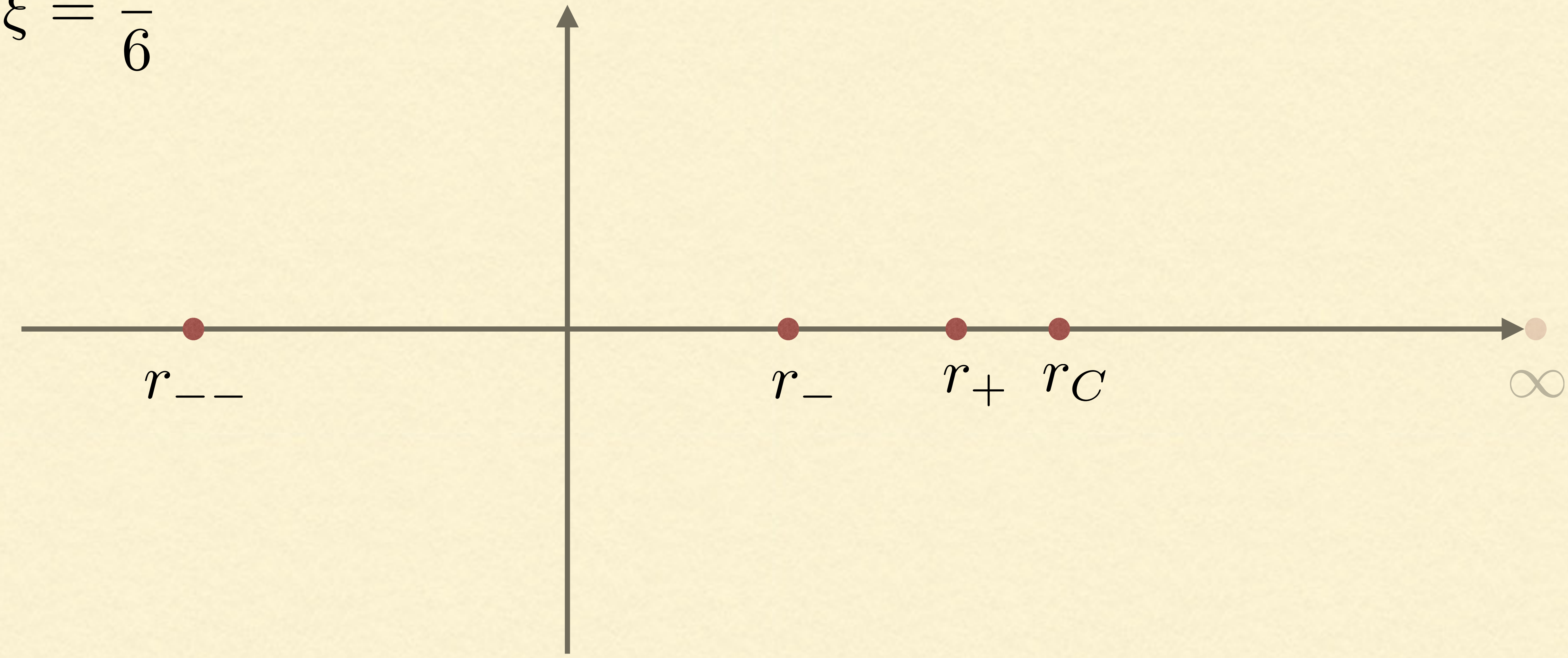
THE BRÜCKER EQUATION



anti-de Sitter

THE HEUN EQUATION

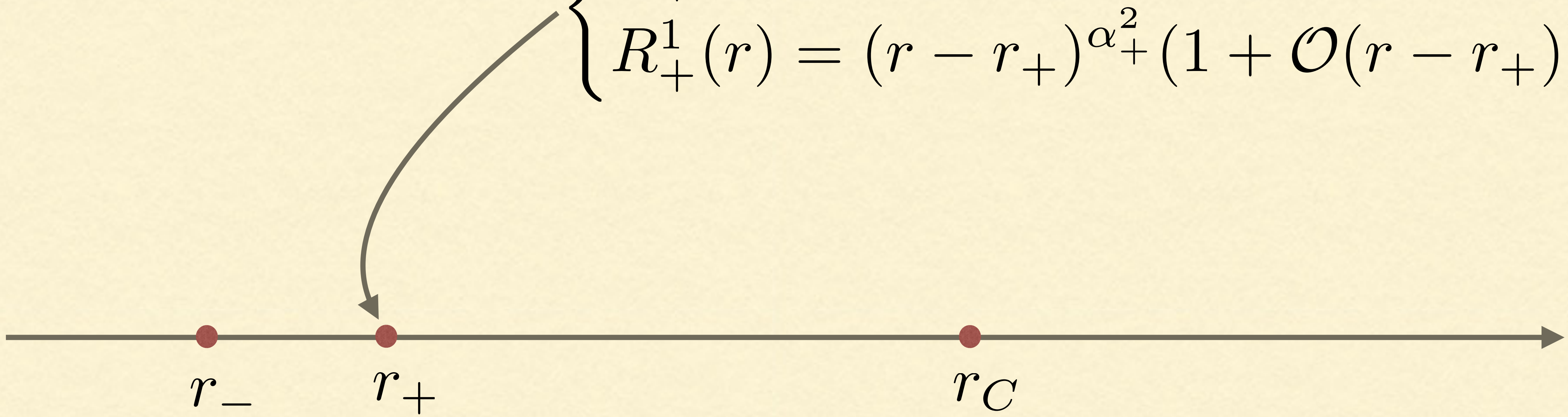
$$\xi = \frac{1}{6}$$



de Sitter

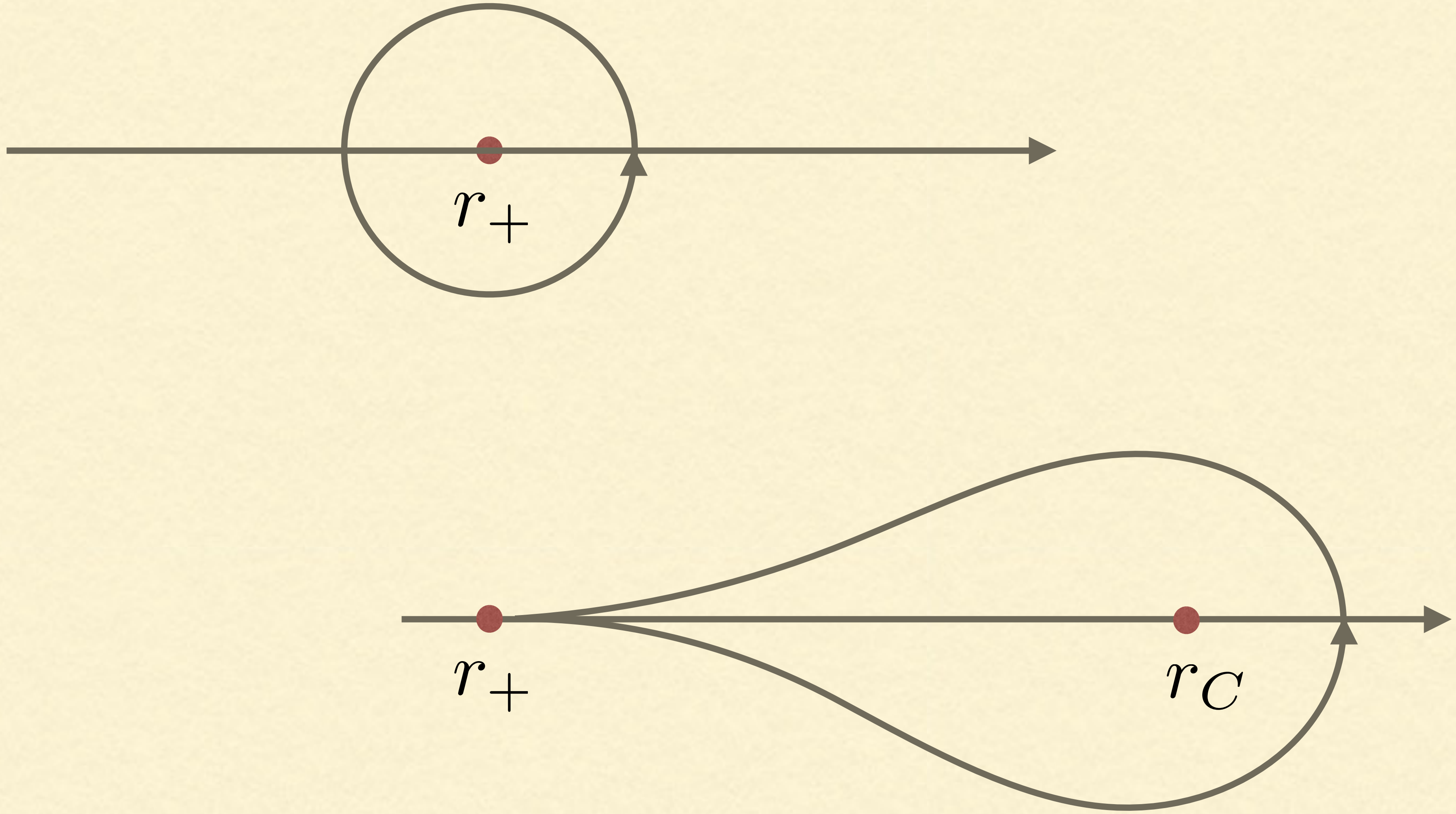
Local solutions: plane waves

$$
\begin{cases}
R^1_+(r) = (r - r_+)^{\alpha^1_+}(1 + \mathcal{O}(r - r_+)) \\
R^1_+(r) = (r - r_+)^{\alpha^2_+}(1 + \mathcal{O}(r - r_+))
\end{cases}
$$



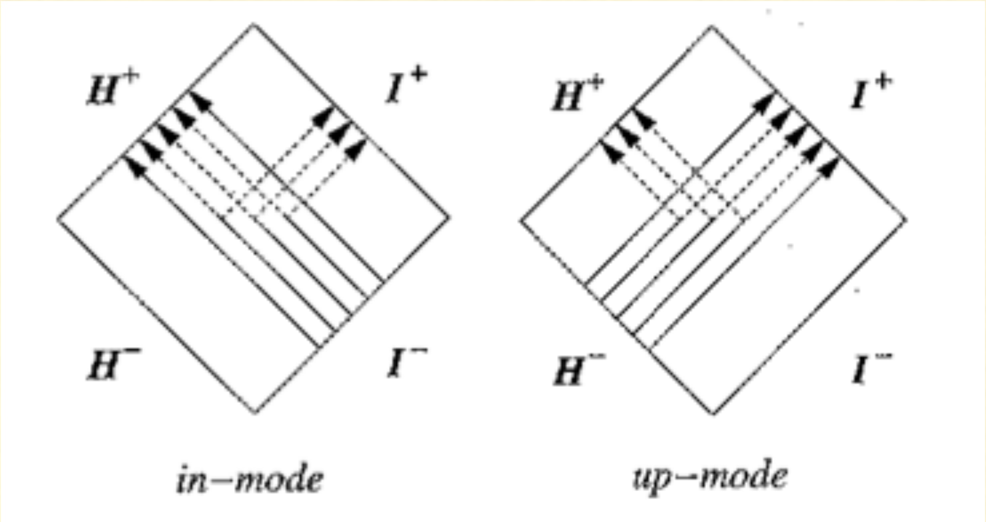
Monodromy

$$R^1_+(r') = e^{2\pi i \alpha^1_+} R^1_+(r)$$
$$R^2_+(r') = e^{2\pi i \alpha^2_+} R^2_+(r)$$



$$R^1_C(r) = E^{11}_{+C} R^1_+(r) + E^{12}_{+C} R^2_+(r)$$
$$R^2_C(r) = E^{21}_{+C} R^1_+(r) + E^{22}_{+C} R^2_+(r)$$

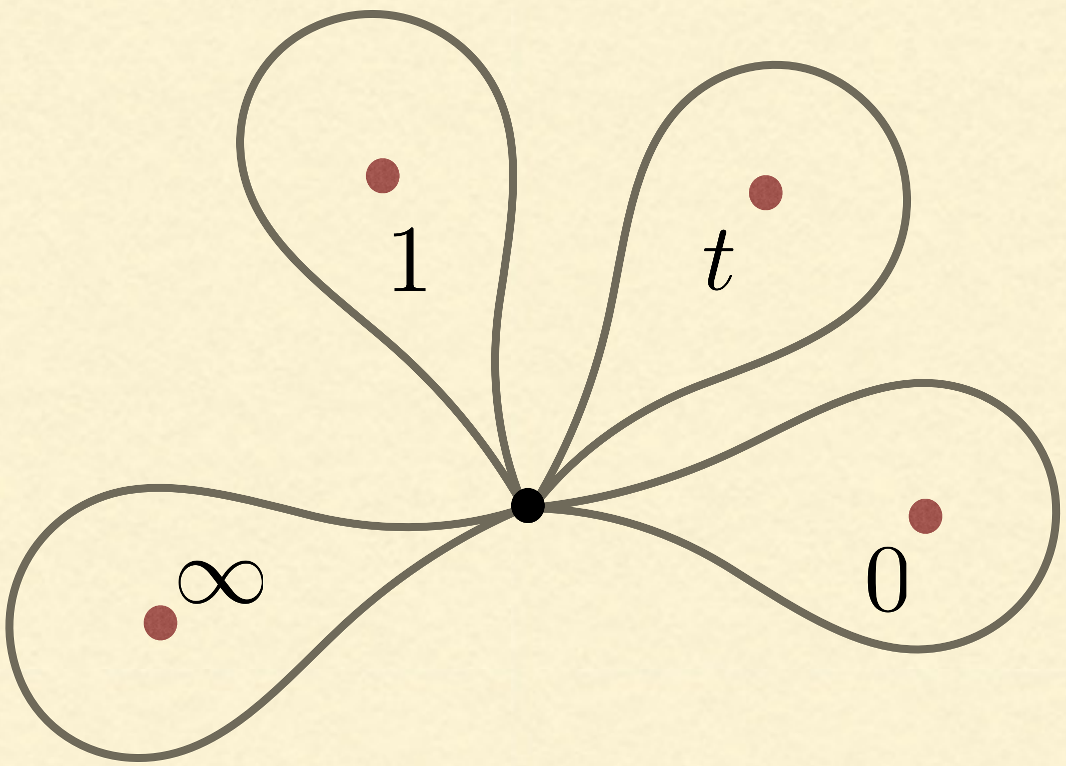
Connection problem (almost) solves scattering problem!



$$u^{\text{in}}_+(r) = \frac{1}{\mathcal{T}} u^{\text{in}}_C(r) + \frac{\mathcal{R}}{\mathcal{T}} u^{\text{out}}_C(r)$$

Extra symmetry (time-reversal) allows for normalization

$$M_{+C} = E_{+C}^{-1} \begin{pmatrix} e^{2\pi i \alpha_C^1} & 0 \\ 0 & e^{2\pi i \alpha_C^2} \end{pmatrix} E_{+C}$$



$$M_\infty M_0 M_t M_1 = \mathbb{1}$$

SU(2) field of 4 monopoles:

$$\frac{d}{dz}\Phi(z) = A(z)\Phi(z) \qquad A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}$$

$$\Phi(z) = \begin{pmatrix} y^{(1)}(z) & y^{(2)}(z) \\ w^{(1)}(z) & w^{(2)}(z) \end{pmatrix}$$

$$A(z) = \left(\frac{d}{dz}\Phi(z)\right)\Phi(z)^{-1}$$

“holomorphic flat connection"

$$F(z) = dA + A \wedge A = 0$$

Many potentials for the same equation!

$$\partial_z^2 y - (\mathrm{Tr}A + \partial_z \log A_{12})\partial_z y + (\det A - \partial_z A_{11} + A_{11}\partial_z \log A_{12})y = 0,$$

$$y'' + p(z,t)y' + q(z,t)y = 0,$$

$$p(z,t) = \frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda},$$

$$q(z,t) = \frac{\kappa_1(\kappa_2+1)}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)},$$

Can embed system into flat holomorphic connection

$$A_z(z,t) = A(z), \qquad A_t(z,t) = -\frac{A_t}{z-t}$$

Isomonodromy deformations!

Riemann, Poincaré, Hilbert: find an ODE with prescribed monodromy data;

Poincaré, Fuchs: ODE has too few parameters! Need matricial system;

Schlesinger: matrices have too many parameters! Families with same monodromy data!

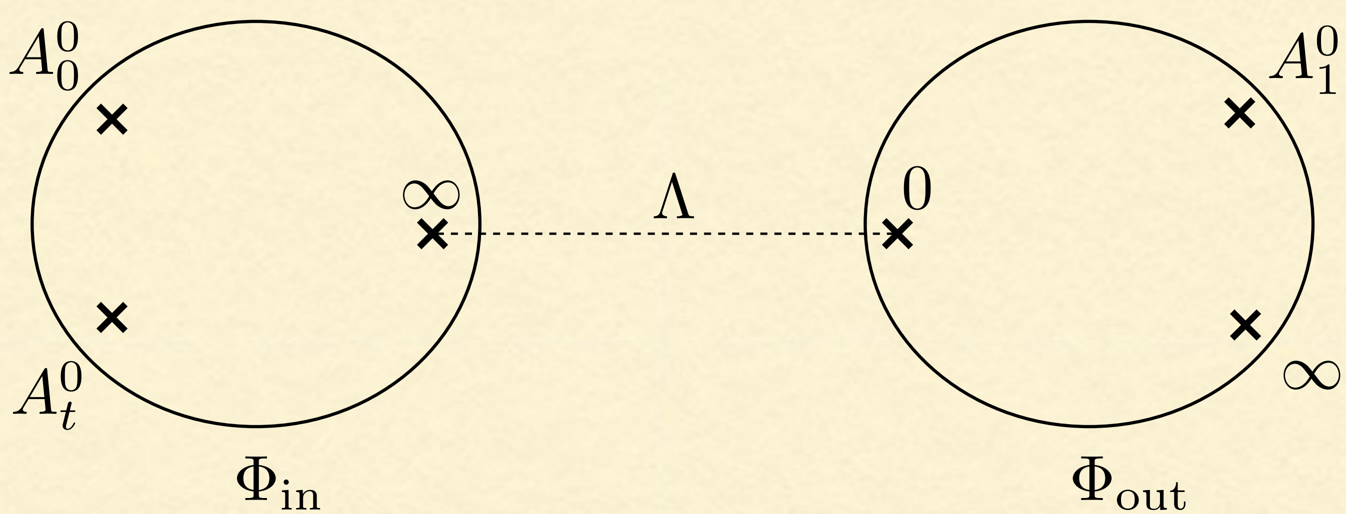
Painlevé, Garnier, Fuchs: what it means to solve ODE?

Miwa, Jimbo, Okamoto: infinite dimensional symmetry (tau functions)

$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_t, A_0], \qquad \frac{\partial A_1}{\partial t} = \frac{1}{t-1}[A_t, A_1],$$

$$\frac{\partial A_t}{\partial t} = \frac{1}{t}[A_0, A_t] + \frac{1}{t-1}[A_1, A_t].$$

Deform to $t \to 0$



Study behavior in terms of composite monodromy

$$\operatorname{Tr}\Lambda = 2\cos\pi\sigma_{0t}$$

Schlesinger equations: Painlevé VI transcendent:

$$\ddot{\lambda} = \frac{1}{2}\left(\frac{1}{\lambda}+ \frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\dot{\lambda}^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\dot{\lambda}$$
$$+\frac{\lambda(\lambda-1)(\lambda-t)}{2t^2(1-t)^2}\left(\theta_\infty^2 - \theta_0^2\frac{t}{\lambda^2}+\theta_1^2\frac{t-1}{(\lambda-1)^2}+\left(1-\theta_t^2\right)\frac{t(t-1)}{(\lambda-t)^2}\right)$$

Actually Hamiltonian system:

$$\frac{d\lambda}{dt} = \{K,\lambda\}, \qquad \frac{d\mu}{dt} = \{K,\mu\}$$

$$K(\lambda,\mu,t) = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[\mu^2 - \left(\frac{\theta_0}{\lambda}+\frac{\theta_1}{\lambda-1}+\frac{\theta_t-1}{\lambda-t}\right)\mu + \frac{\kappa_1(\kappa_2+1)}{\lambda(\lambda-1)}\right]$$

“Effective potential”: tau function

$$\frac{d}{dt}\log\tau(t,\{\theta_i\}) = \frac{1}{t}\mathrm{Tr}(A_0A_t) + \frac{1}{t-1}\mathrm{Tr}(A_1A_t)$$

Initial value problem for tau:

$$\left.\frac{d}{dt}\log\tau(t,\{\theta_i\})\right|_{t=t_0} = \frac{\theta_0\theta_t}{t_0} + \frac{\theta_1\theta_t}{t_0-1} + K_0$$

$$\left.\frac{d^2}{dt^2}\log\tau(t,\{\theta_i\})\right|_{t=t_0} = -\frac{\theta_0\theta_t}{t_0^2} - \frac{\theta_1\theta_t}{(t_0-1)^2} + \frac{\kappa_1\theta_t}{t_0(t_0-1)} - \frac{2t_0-1}{t_0(t_0-1)}K_0$$

Formally can be inverted to give monodromy data in terms of ODE parameters

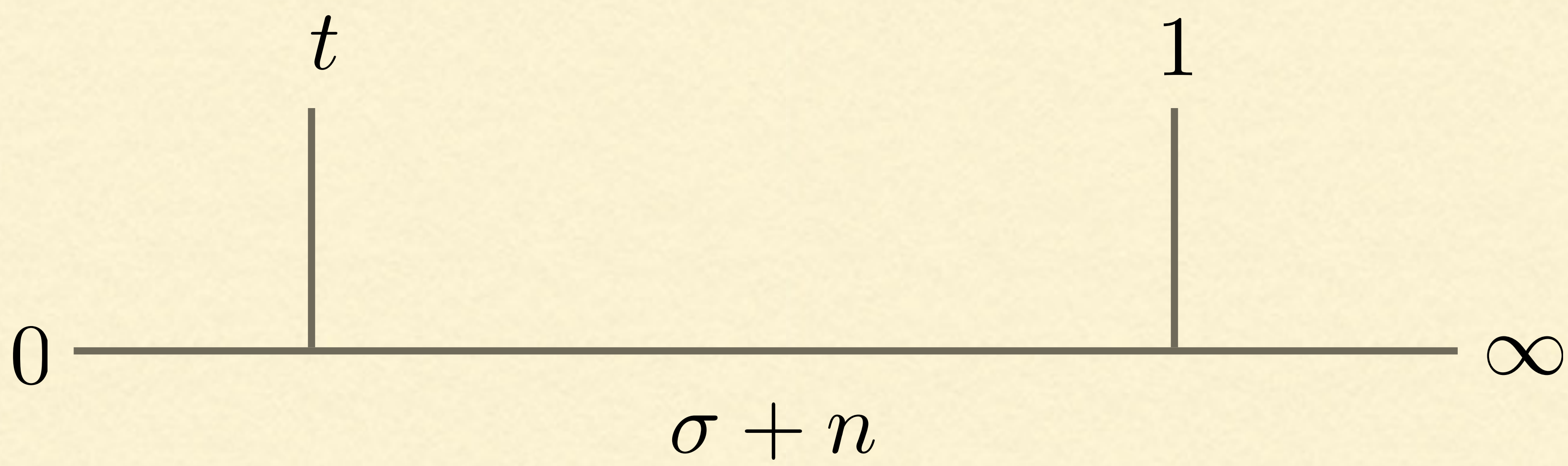
Liouville's level 2 null field:

$$(\partial_z^2 + \frac{\gamma^2}{2} T_{zz}) e^{-1/2\gamma\phi(z,\bar{z})} = 0$$

Background with 4 "conical" singularities:

$\exp[\theta_i/\gamma\phi]$ $\qquad h_i = \frac{1}{4}(1-(1-\theta_i)^2)$

conformal blocks



tau function gives insights for intermediate states in CFT

Expansions for tau exist following the proof of AGT conjecture

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\left(1 + \mathcal{B}_1(\boldsymbol{\theta},\sigma)t + \mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2 \ldots\right) +$$
$$+ C_{\pm 1}\, t^{\Delta_{\sigma \pm 1} - \Delta_0 - \Delta_t}\left(1 + \mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma)t + \ldots\right) + C_{\pm 2} t^{\Delta_{\sigma \pm 2} - \Delta_0 - \Delta_t}\left(1 + \ldots\right)$$

with

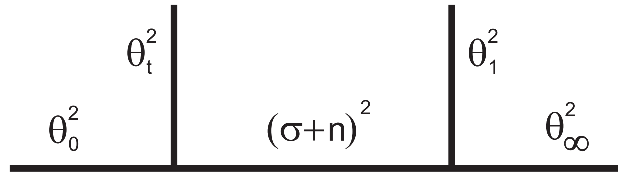
$$\mathcal{B}_1(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

$$\mathcal{B}_2(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_0 + \Delta_t + 1)(\Delta_\sigma - \Delta_\infty + \Delta_1)(\Delta_\sigma - \Delta_\infty + \Delta_1 + 1)}{4\Delta_\sigma(2\Delta_\sigma + 1)}$$
$$- \frac{\left[(1 + 2\Delta_\sigma)(\Delta_0 + \Delta_t) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_0 - \Delta_t)^2\right]\left[(1 + 2\Delta_\sigma)(\Delta_\infty + \Delta_1) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_\infty - \Delta_1)^2\right]}{2(2\Delta_\sigma + 1)(4\Delta_\sigma - 1)^2},$$

$$\mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma) = \mathcal{B}_1(\boldsymbol{\theta},\sigma \pm 1).$$

Observation. PVI tau function is a linear combination of $c = 1$ conformal blocks:

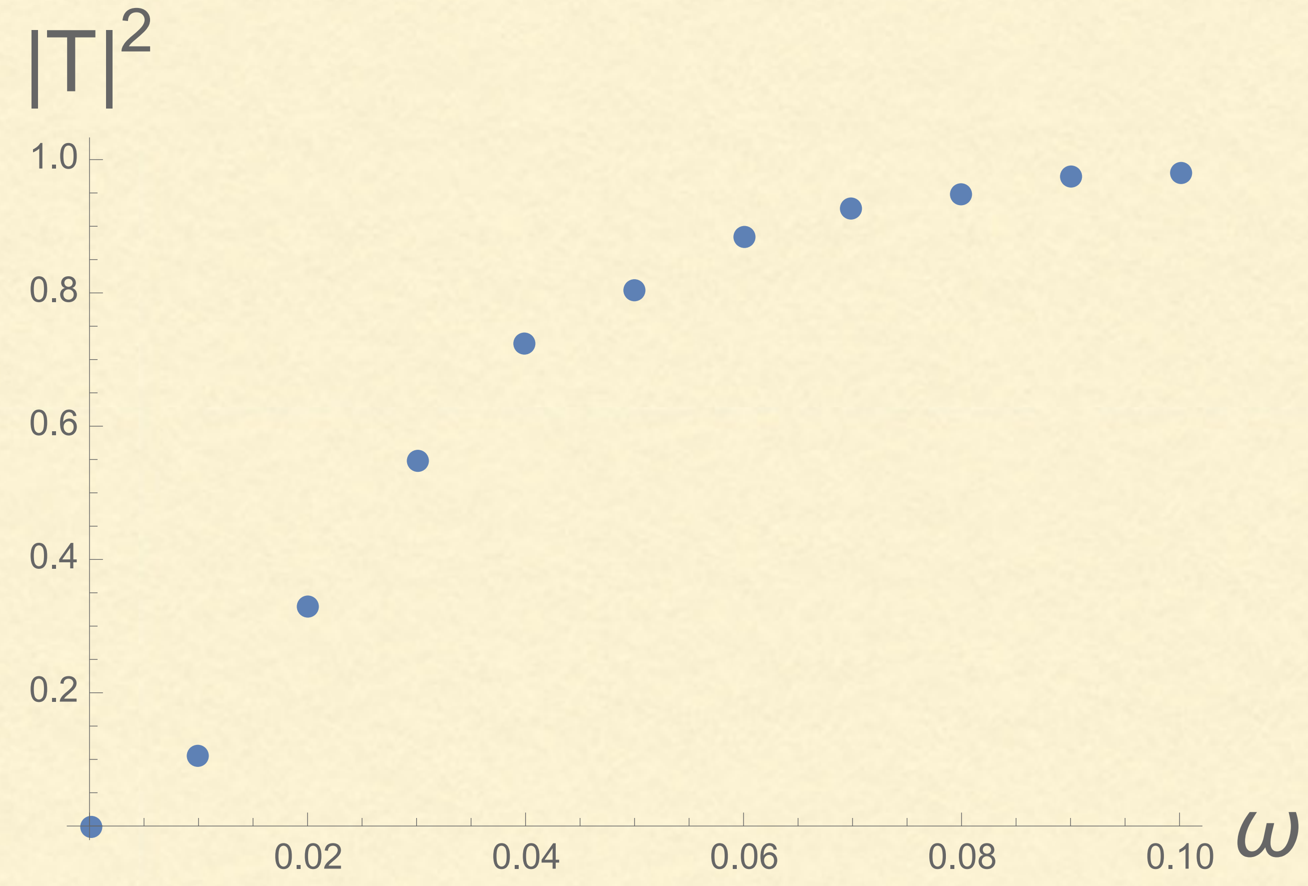
$$\tau(t) = \sum_{n \in \mathbb{Z}} C_n t^{\Delta_{\sigma+n} - \Delta_0 - \Delta_t} \mathcal{B}(\boldsymbol{\theta}, \sigma + n, t)$$

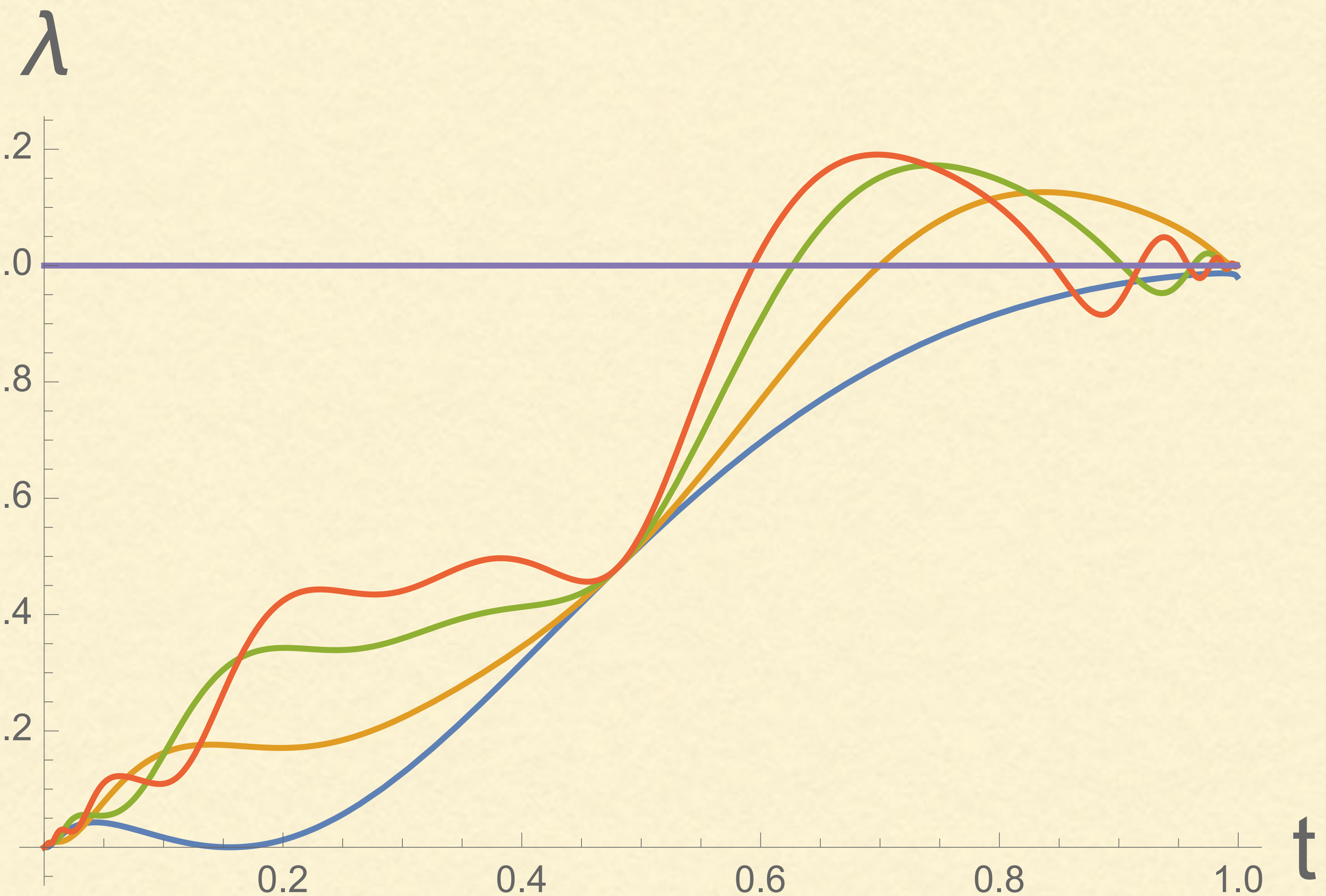
Graphical representation of $\mathcal{B}(\boldsymbol{\theta}, \sigma + n, t)$: 

Gamayun, Iorgov, Lisovyy 1207.0787 1302.1832

Formal expressions

$$|\mathcal{T}|^2 = \left| \frac{2 \sin 2\pi\theta_i \sin 2\pi\theta_j}{\cos 2\pi(\theta_i - \theta_j) - \cos \pi\sigma_{ij}} \right|$$

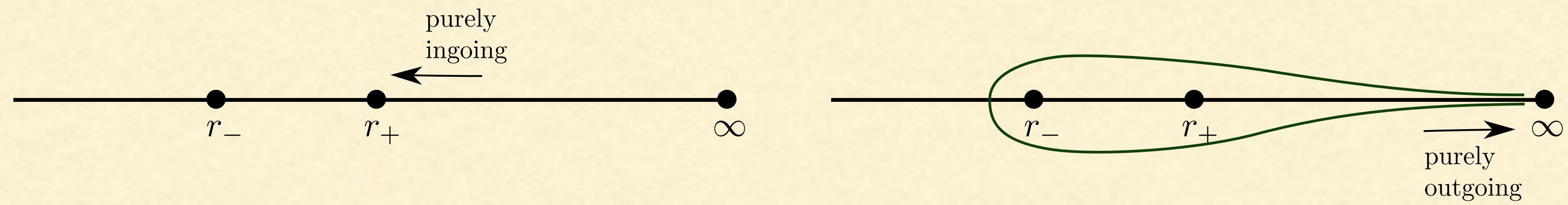


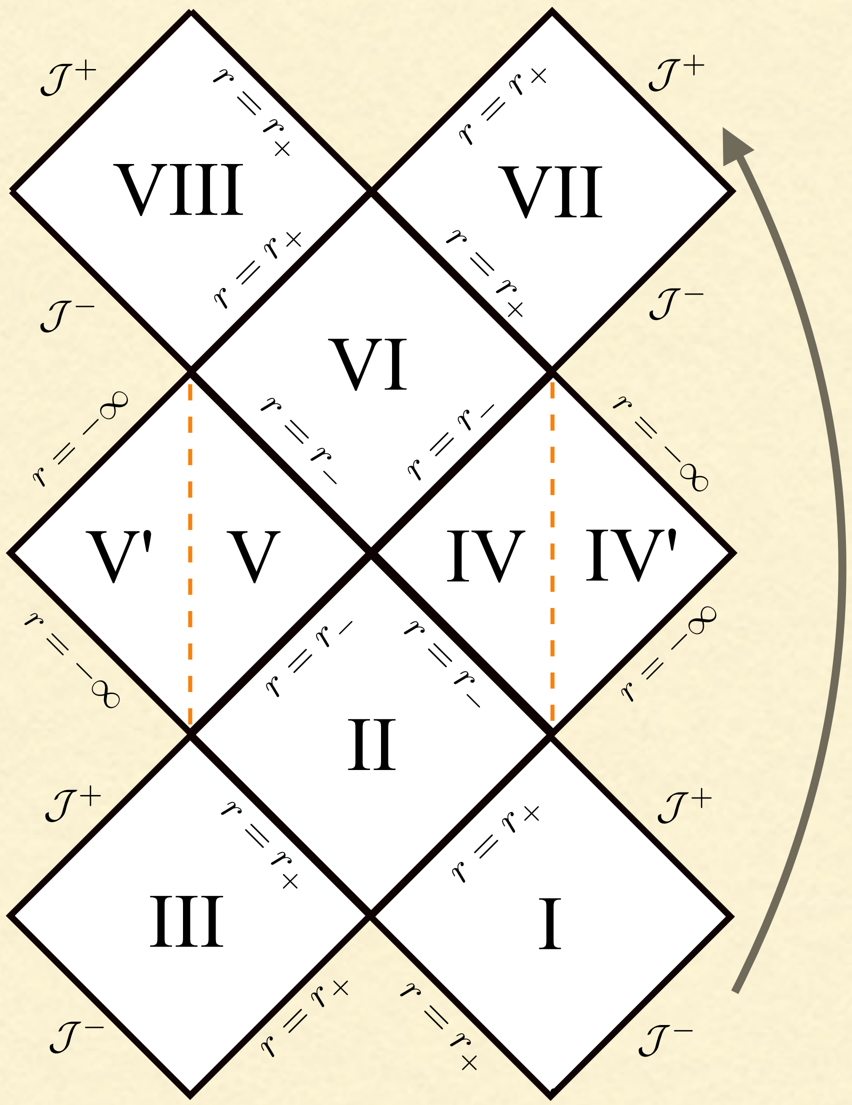


Simple expressions in the near-extremal case

- Exact expressions in terms of PVI tau-function
- Near extremal expressions in terms of elementary functions
- Survey for greybody factors in progress
- Teukolsky: quasi-normal modes
- Aretaxis' instabilities?

Scattering between different asymptotic regions





$$\mathrm{Tr}(M_+ M_-)^n$$

Sum over different n:
illusion of unitarity!

Important applications in adS/CFT

Kerr and Painlevé V

- Same history, confluent Heun. Isomonodromy also gives solution for scattering in terms of Painlevé V tau:

$$\partial_r(Q(r)\partial_r R(r)) + \left(-C_{\ell,m} + \frac{W_r^2}{Q(r)}\right) R(r) = 0 \; ,$$

$$Q(r) = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-),$$
$$W_r = \omega(r^2 + a^2) - am.$$

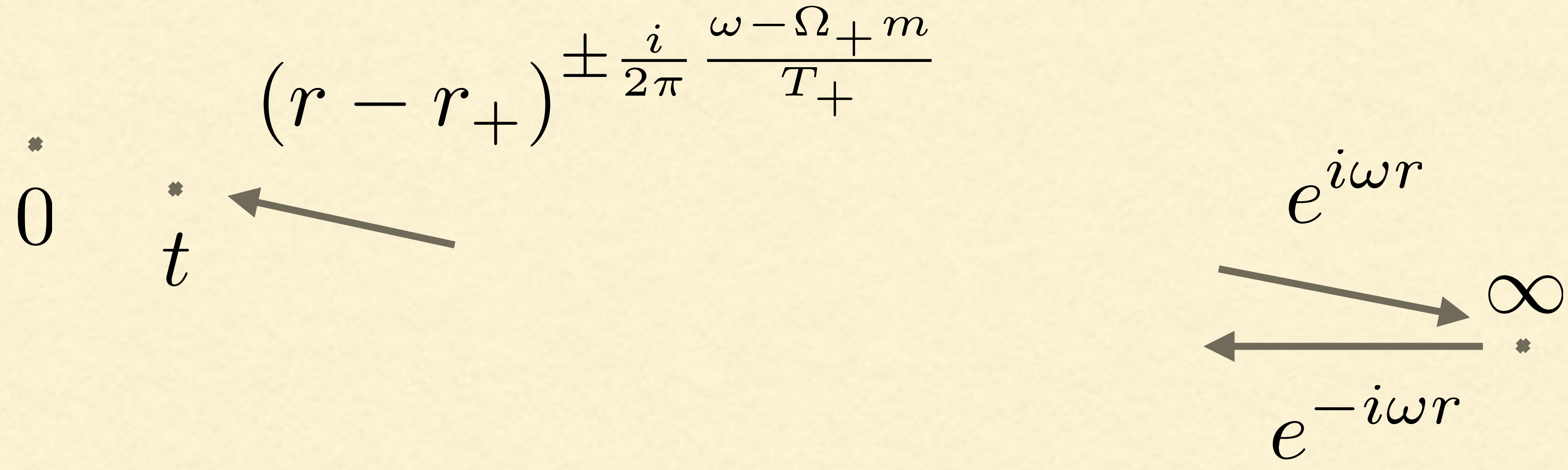
$$z = 2i\omega(r - r_-) \qquad y(z) = (r - r_-)^{\theta_0/2}(r - r_+)^{\theta_{t_0}/2}R(r)$$

$$t_0 = 2i\omega(r_+ - r_-)$$

Confluent Heun (standard form):

$$\frac{d^2y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y(z) = 0,$$

$$p(z) = \frac{1 - \theta_{t_0}}{z - t_0} + \frac{1 - \theta_0}{z}, \quad q(z) = -\frac{1}{4} + \frac{c_0}{z} + \frac{c_{t_0}}{z - t_0},$$



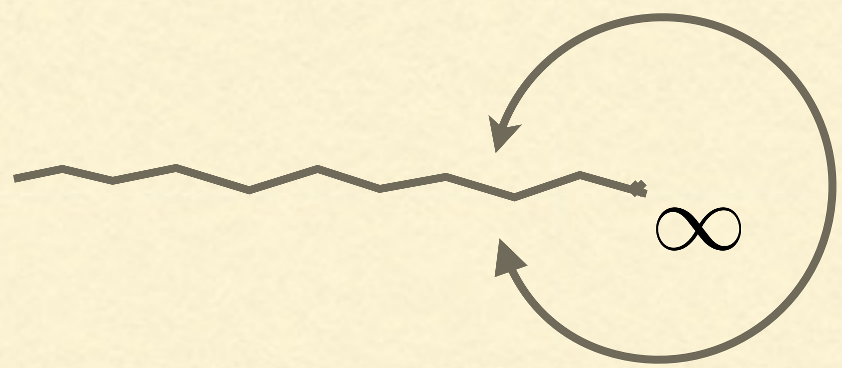
$$j = -i\, Q(r)[R(r)^* \partial_r R(r) - R(r) \partial_r R(r)^*].$$

$$R^-_{t_0} = \frac{1}{\mathcal{T}} R^+_\infty + \frac{\mathcal{R}}{\mathcal{T}} R^-_\infty,$$

$$|\mathcal{T}|^2 = \left| \frac{2 \sin \pi\theta \sin \pi\theta_{t_0}}{C^{-1/2} \cos \pi\theta_0 + \cos \pi(\theta - \theta_{t_0})} \right|,$$

$$C = 1 + s_1 s_2 \qquad \theta = \theta_\infty + (i\pi)^{-1} \log C$$

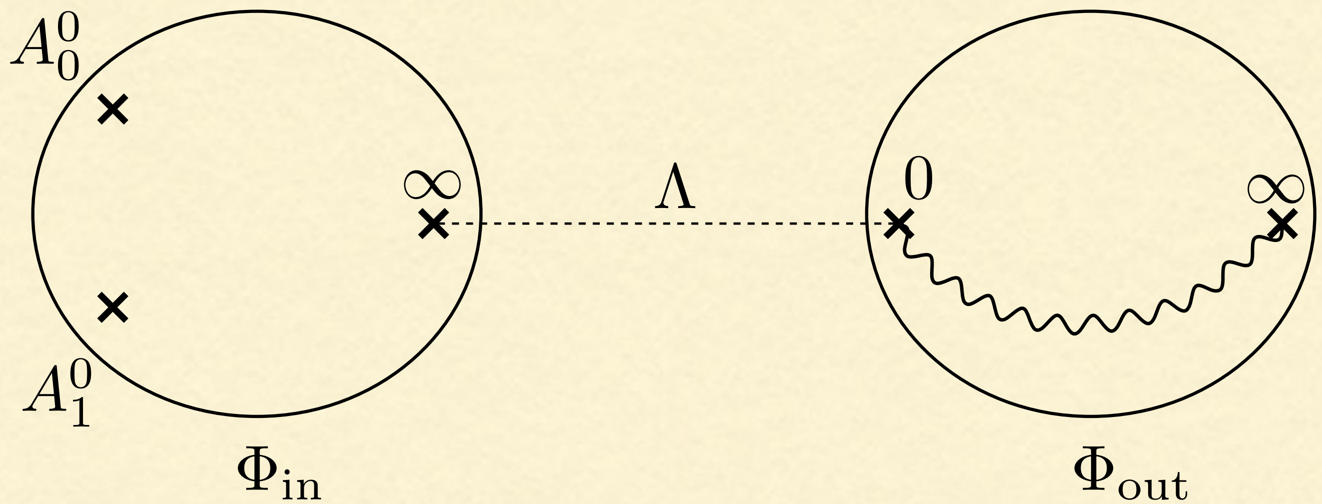
Stokes phenomenon near infinity:



$$S_{2j} = \begin{pmatrix} 1 & s_{2j} \\ 0 & 1 \end{pmatrix}, \quad S_{2j+1} = \begin{pmatrix} 1 & 0 \\ s_{2j+1} & 1 \end{pmatrix};$$

$$M_\infty\big|_{\mathcal{S}_j} = S_j S_{j+1} e^{\pi i \theta_\infty \sigma_3}.$$

$$M_\infty M_t M_0 = Id$$



$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_t, A_0],$$
$$\frac{\partial A_t}{\partial t} = -\frac{1}{t}[A_t, A_0] - \frac{1}{2}[A_t, \sigma_3].$$

Invariant monodromy data $\{\theta_i\}, s_1, s_2$

$$\frac{d^2y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y = 0,$$

$$p(z) = \frac{1-\theta_0}{z} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda},$$

$$q(z) = -\frac{1}{4} + \frac{C_0}{z} + \frac{C_t}{z-t} + \frac{\mu}{z-\lambda},$$

$$\mu^2 - \left[\frac{\theta_0 - 1}{\lambda} + \frac{\theta_t - 1}{\lambda - t}\right]\mu + \frac{C_0}{\lambda} + \frac{C_t}{\lambda - t} = \frac{1}{4}.$$

NOT Hamilton-Jacobi, tau function!

$$\frac{d}{dt}\log\tau(t,\{\theta_i\},s_1,s_2) = -\frac{1}{2}\mathrm{Tr}\sigma_3 A_t - \frac{1}{t}\mathrm{Tr}A_0 A_t$$

$$\frac{d}{dt}\log\tau_V(t)\Big|_{t=t_0} = c_{t_0} - \frac{\theta_0(\theta_{t_0}-1)}{t_0},$$

$$\frac{d^2}{dt^2}\log\tau_V(t)\Big|_{t=t_0} = -\frac{c_{t_0}}{t_0} + \frac{\theta_{t_0}-1}{2t_0} + \frac{\theta_0(\theta_{t_0}-1)}{t_0^2},$$

Also solved by AGT instantons, irregular CFT blocks

SUMMARY

- Exact expressions in terms of new transcendental functions
- Same integrable structure as Einstein's
- Obvious extensions: higher spin and Teukolsky
- Experimental predictions: GRB, superradiance
- New tools for stability and AdS/CFT - stability studies

THANK YOU!

PROPESQ

Pró-Reitoria para Assuntos de Pesquisa e Pós-Graduação