On the symmetries of Yang-Mills squared

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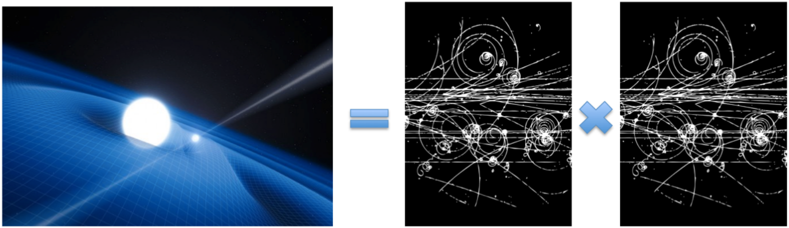
Introduction

Gravity as the square of Yang-Mills



Bern-Carrasco-Johansson colour-kinematic duality

- Color-dressed n-point tree amplitude of Yang-Mills theory:

$$A_n^{\text{tree}} = \sum_{i \in \text{trivalent graphs}} \frac{c_i n_i}{\prod_{a_i} p_{a_i}^2}$$

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- For triples of diagrams s.t.

$$c_1 + c_2 + c_3 = 0 \quad \Rightarrow \quad \underbrace{n_i \to n_i + s_i \Delta}_{\text{generalised gauge transformations}}$$

$$\boxed{\begin{array}{ccc} c_i = -c_j & \Leftrightarrow & n_i = -n_j \\ c_i + c_j + c_k = 0 & \Leftrightarrow & n_i + n_j + n_k = 0 \end{array}}$$

[Bern-Dennen-Huang-Kiermaier:2010]

- Conjectured to hold at loop level! (Beyond KLT relations)
  [Bern-Carrasco-Johansson:2008, 2010]

BCJ double-copy relations

Kinematic factor

Colour factor

Loops

Kinematic factors

$$\underbrace{\sum_i \int \prod^L \frac{d^D p_l}{(2\pi)^D S_i} \frac{c_i n_i}{\prod_{a_i} p_{a_i}^2}}_{\text{gauge theory amplitude}} \longrightarrow \underbrace{\sum_i \int \prod^L \frac{d^D p_l}{(2\pi)^D S_i} \frac{\tilde{n}_i n_i}{\prod_{a_i} p_{a_i}^2}}_{\text{gravity amplitude}}$$

Diagrams

Propagators

$$c_i \rightarrow \tilde{n}_i$$

[Bern-Dennen-Huang-Kiermaier:2010]

BCJ double-copy relations

- Conceptually compelling and computationally powerful:

$$\mathcal{N} = 8 \ \text{ supergravity four-point to 4 loops! (finite)}$$

[Bern-Carrasco-Dixon-Johansson-Roiban:2009]

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- Can be explained by supersymmetry and $E_{7(7)}$ U-duality [Bjornsson-Green: 2010, Bossard-Howe-Stelle:2011; Elvang-Freedman-Kiermaier:2011, Bossard-Howe-Stelle-Vanhove:2011]

- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" (Bjornsson and Green)

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- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" (Bjornsson and Green)

- $D = 4, \mathcal{N} = 5$ supergravity finite to 4 loops, contrary to expectations:

"Enhanced" cancellations

[Bern-Davies-Dennen:2014]

Non-perturbative relations

Double-copy relations between classical solutions:

- Kerr-Schild spacetimes: $g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu$

- Can be obtained as the "double copy" of Yang-Mills solutions:

$$A_\mu^a = \hat{k}_\mu \phi^a$$

- Example: Double copy of Coulomb solution for the superposition of static colour charge $\rightarrow$ Schwarzschild black hole

[Monteiro-O'Connell-White:2014]

Questions

To what extent, or in what sense, can one regard gravity as the square of Yang-Mills?

- Can we understand the origin of BCJ duality?
- Can we go beyond amplitudes?
- Can we relate classical solutions? [Monteiro-O'Connell-White:2014]
- Is there something to say beyond perturbation theory?

Gravity=gauge×gauge dictionary

- “Going on-shell” $\longrightarrow$ new relations between gravity and gauge theories
- Mysterious: can we climb back down?

Gravity=gauge×gauge dictionary

- "Going on-shell" $\longrightarrow$ new relations between gravity and gauge theories
- Mysterious: can we climb back down?
- Build covariant fields of (super)gravity from those of (super) Yang-Mills
- Consistency check: symmetries

Tensoring states vs fields

Much of the squaring literature invokes a mysterious product:

$$A_\mu(x) \otimes \tilde{A}_\nu(x)$$

[Siegel:1988, 1995]

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- Arbitrary non-Abelian gauge groups G_L and G_R: where do the gauge indices go?

- Does it obey the Leibnitz rule

$$\partial_\mu(f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$

The off-shell product

$G_L \times G_R$ product rule within field theory which is valid whether or not there is an underlying string interpretation:

$$\boxed{f \otimes g := f \star \Phi \star g}$$

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- $\star$ denotes a convolutive tensor product with Killing form:

$$[f \star g](x) = \int d^D y \langle f(y) \otimes g(x-y) \rangle$$

- Φ is the "spectator" $G_L \times G_R$ bi-adjoint scalar field

Spectator scalar field

- Cachazo-He-Yuan unified scattering formulae:

$$\underbrace{(C_L N_L)}_{\text{left gauge } G_L} \times \underbrace{(C_R N_R)}_{\text{right gauge } G_R} \rightarrow \underbrace{(C_L C_R)}_{G_L \times G_R \text{ scalar}} \times \underbrace{(N_L N_R)}_{\text{gravity}}$$

A common form for spin 0, 1, 2 tree-level scattering in any dimension [Cachazo-He-Yuan:2013]

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- Suggests a modified relation:

$$\boxed{gauge \times gauge = \phi^3 \times gravity}$$

[Hodges:2011, 2012]

(Ambi)twistor-strings [Mason-Skinner:2013, Geyer-Lipstein-Mason:2013,2014]

$\mathcal{N} = 1$ supergravity from the product of Yang-Mills

Equipped with $\star$ and Φ let's consider a simple example at linearized level:

$D = 4$, $[\mathcal{N} = 1 \quad SYM] \otimes [\mathcal{N} = 0 \quad YM]$

- $(4+4)$ off-shell $\mathcal{N}_L = 1$ G_L Yang-Mills multiplet:

$$A_\mu, \qquad \chi, \qquad D$$

- $(3+0)$ off-shell $\mathcal{N}_R = 0$ G_R Yang-Mills multiplet:

$$\tilde{A}_\nu$$

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- Yields $(12 + 12)$ new-minimal $\mathcal{N} = 1$ supergravity:

$$g_{\mu\nu}, \qquad B_{\mu\nu}, \qquad \psi_\mu, \qquad V_\mu$$

  with general covariance, 2-form gauge invariance, local supersymmetry and local chiral symmetry [Sohnius-West:1981]

Left super Yang-Mills multiplet

- Left real superfield

$$V(x,\theta) \sim C + i\bar{\theta}\zeta + i\theta^2 F - \theta\gamma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\left(\chi + \partial\!\!\!/\zeta\right) + \theta^4\left(D + \Box C\right)$$

- Transforming under local supergauge and global super-Poincaré:

$$\delta V = \underbrace{\Lambda + \bar{\Lambda}}_{\text{local Abelian supergauge}} + \overbrace{[V, X]}^{\text{global non-Abelian } G_L} + \underbrace{\delta_{(a,\lambda,\epsilon)}V}_{\text{global super-Poincaré}}$$

- $\Lambda(x,\theta)$ chiral superfield of supergauge parameters

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- Wess-Zumino gauge $V \to (A, \chi, D)$

$$V|_{WZ} = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\chi} - i\bar{\theta}^2\theta\chi + \frac{1}{2}\bar{\theta}^2\theta^2 D$$

Right Yang-Mills multiplet

- The right Yang-Mills field $\tilde{A}_\nu$ transforms as

$$\delta \tilde{A}_\nu = \underbrace{\partial_\nu \tilde{\lambda}}_{\text{local Abelian gauge}} + \overbrace{[\tilde{A}_\nu, \tilde{X}]}^{\text{global non-Abelian } G_R} + \underbrace{\delta_{(a,\lambda)} A_\nu}_{\text{global Poincaré}}$$

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- The spectator bi-adjoint scalar Φ field transforms as

$$\delta\Phi = \overbrace{-[\Phi, X] - [\Phi, \tilde{X}]}^{\text{global non-Abelian } G_L \times G_R} + \underbrace{\delta_a\Phi}_{\text{global Poincaré}}$$

$[SYM] \times [YM]$ dictionary

The gravitational symmetries are reproduced here from those of Yang-Mills by invoking the gravity/Yang-Mills dictionary:

Fields	φ_ν	$=$	V	$\star$	Φ	$\star$	$\tilde{A}_\nu$	*real superfield*
Paras	ϕ	$=$	V	$\star$	Φ	$\star$	$\tilde{\lambda}$	*real superfield*
	S_ν	$=$	Λ	$\star$	Φ	$\star$	$\tilde{A}_\nu$	*chiral superfield*

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Variation: $\delta\varphi_\nu = \delta V \star \Phi \star \tilde{A}_\nu + V \star \delta\Phi \star \tilde{A}_\nu + V \star \Phi \star \delta\tilde{A}_\nu$

$$\boxed{\delta\varphi_\nu = S_\nu + \bar{S}_\nu + \partial_\nu\phi + \delta_{(a,\lambda,\epsilon)}\varphi_\nu}$$

$$\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle \qquad \partial_\mu(f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g)$$

$\mathcal{N} = 1$ new-mininal supergravity

New-minimal formulation of $\mathcal{N} = 1$ supergravity:

$$\begin{aligned}\varphi_\nu(x,\theta,\bar\theta) =& C_\nu + i\theta\chi_\nu - i\bar\theta\bar\chi_\nu + i\theta^2 F_\nu - i\bar\theta^2\bar F_\nu - \theta\sigma^\mu\bar\theta(g_{\mu\nu} + B_{\mu\nu}) \\ &+ i\theta^2\bar\theta\left(\bar\psi_\nu + \frac{i}{2}\bar\sigma^\rho\partial_\rho\chi_\nu\right) - i\bar\theta^2\theta\left(\psi_\nu + \frac{i}{2}\sigma^\rho\partial_\rho\bar\chi_\nu\right) \\ &+ \frac{1}{2}\bar\theta^2\theta^2\left(V_\nu + \frac{1}{2}\Box C_\nu\right)\end{aligned}$$

At linearised level $\varphi_\nu(x,\theta,\bar\theta)$ under local supergauge transformations:

$$\delta\varphi_\mu = S_\mu + \bar S_\mu + \partial_\mu\phi + \delta_{(a,\lambda,\epsilon)}\varphi_\nu$$

[Cecotti et al:1987, Ferrara et al:1988]

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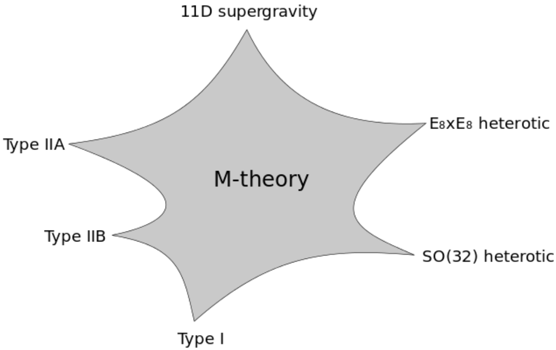
- The local gravitational symmetries of general covariance, 2-form gauge invariance, local supersymmetry and local chiral symmetry follow from those of Yang-Mills at linear level

U-duality



- Duff, Khuri, Strominger, Sen, Witten. . . many others
- U-duality [Hull-Townsend:1994]

U-duality

- U-duality: supergravities are characterized by cosets G/H [Cremmer-Julia:1979]

n-torus	U-duality	G	H
1	$SO(1,1,\mathbb{Z})$	$SO(1,1,\mathbb{R})$	–
2	$SL(2,\mathbb{Z}) \times SO(1,1,\mathbb{Z})$	$SL(2,\mathbb{R}) \times SO(1,1,\mathbb{R})$	$SO(2,\mathbb{R})$
3	$SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$	$SL(2,\mathbb{R}) \times SL(3,\mathbb{R})$	$SO(2,\mathbb{R}) \times SO(3,\mathbb{R})$
4	$SL(5,\mathbb{Z})$	$SL(5,\mathbb{R})$	$SO(5,\mathbb{R})$
5	$SO(5,5,\mathbb{Z})$	$SO(5,5,\mathbb{R})$	$SO(5,\mathbb{R}) \times SO(5,\mathbb{R})$
6	$E_{6(6)}(\mathbb{Z})$	$E_{6(6)}(\mathbb{R})$	$USP(8)$
7	$E_{7(7)}(\mathbb{Z})$	$E_{7(7)}(\mathbb{R})$	$SU(8)$
8	$E_{8(8)}(\mathbb{Z})$	$E_{8(8)}(\mathbb{R})$	$SO(16,\mathbb{R})$

- $\rightarrow$ Consider $D = 3$, $\mathcal{N} = 1, 2, 4, 8$-extended Yang-Mills theories

- $D = 3$ amplitude relations [Bargheer-He-McLoughlin:2012; Lipstein-Mason:2012]
- U-dualities and amplitudes also considered in [Bianchi-Elvang-Freedman:2008; Chiodaroli-Gunaydin-Roiban:2012; Carrasco-Chiodaroli-Gunaydin-Roiban:2013]

Magic Symmetries

$$[\mathcal{N}_L \quad SYM] \otimes [\mathcal{N}_R \quad SYM] \rightarrow [\mathcal{N}_L + \mathcal{N}_R sugra]$$

$\otimes$	$\mathcal{N}_R = 1$	$\mathcal{N}_R = 2$	$\mathcal{N}_R = 4$	$\mathcal{N}_R = 8$
$\mathcal{N}_L = 1$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{su}(2,1)$	$\mathfrak{sp}(4,2)$	$\mathfrak{f}_{4(-20)}$
$\mathcal{N}_L = 2$	$\mathfrak{su}(2,1)$	$\mathfrak{su}(2,1) \times \mathfrak{su}(2,1)$	$\mathfrak{su}(4,2)$	$\mathfrak{e}_{6(-14)}$
$\mathcal{N}_L = 4$	$\mathfrak{sp}(4,2)$	$\mathfrak{su}(4,2)$	$\mathfrak{so}(8,4)$	$\mathfrak{e}_{7(-5)}$
$\mathcal{N}_L = 8$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(8)}$

Freudenthal-Rozenfeld-Tits magic square '55

[LB-Duff-Hughes-Nagy:2013]

Sugra magic squares: [Gunaydin-Sierre-Townsend:1985, Cacciatori-Cerchiai-Marrani:2013]

Division algebras and the magic square

Definition

- An algebra $\mathbb{A}$ defined over the reals $\mathbb{R}$ is said to be *composition* if it has a non-degenerate quadratic form $n : \mathbb{A} \to \mathbb{R}$ such that for $a, b \in \mathbb{A}$,

$$n(ab) = n(a)n(b), \quad \forall a, b \in \mathbb{A},$$

- If n is pos-def then $\mathbb{A}$ is a normed division algebra:

$$ab = 0 \Rightarrow \quad a = 0 \quad \text{or} \quad b = 0.$$

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Hurwitz's theorem: There are only 4 normed division algebras

$\mathbb{A}$	*Construction*	*Dim*	*Division*	*Associative*	*Commutative*	*Ordered*
$\mathbb{R}$	$\mathbb{R}$	1	*yes*	*yes*	*yes*	*yes*
$\mathbb{C}$	$\mathbb{R} \oplus e_1\mathbb{R}$	2	*yes*	*yes*	*yes*	*no*
$\mathbb{H}$	$\mathbb{C} \oplus e_2\mathbb{C}$	4	*yes*	*yes*	*no*	*no*
$\mathbb{O}$	$\mathbb{H} \oplus e_3\mathbb{H}$	8	*yes*	*no*	*no*	*no*
$\mathbb{S}$	$\mathbb{O} \oplus e_8\mathbb{O}$	16	*no*	*no*	*no*	*no*

Division algebras

The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings.

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But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative.

[Baez:2002]

The octonions

- An element $x \in \mathbb{O}$ may be written

$$x = x^a e_a, \qquad a = 0, \ldots, 7, \quad x^a \in \mathbb{R}$$

- One real $e_0 = 1$ and seven $e_i, i = 1, \ldots, 7,$ imaginary elements

$$e_0^* = e_0 \quad \text{and} \quad e_i^* = -e_i$$

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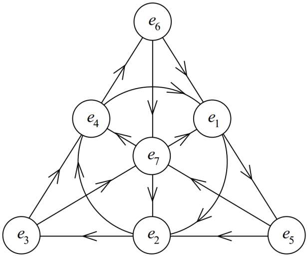
- The octonionic multiplication rule is,

$$e_a e_b = \left(\delta_{a0}\delta_{bc} + \delta_{0b}\delta_{ac} - \delta_{ab}\delta_{0c} + C_{abc}\right) e_c ,$$

- C_{abc} is totally antisymmetric such that

$$C_{0bc} = 0$$

- The non-zero C_{ijk} are given by the Fano plane (the projective plane over $\mathbb{F}_2$)



Division algebras and the magic square

- Isometry algebras of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ projective spaces:

$$\mathfrak{Isom}(\mathbb{RP}^n) \cong \mathfrak{so}(n+1), \quad \mathfrak{Isom}(\mathbb{CP}^n) \cong \mathfrak{su}(n+1), \quad \mathfrak{Isom}(\mathbb{HP}^n) \cong \mathfrak{sp}(n+1)$$

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  $$\mathfrak{Isom}(\mathbb{R}\mathbb{P}^n) \cong \mathfrak{so}(n{+}1), \quad \mathfrak{Isom}(\mathbb{C}\mathbb{P}^n) \cong \mathfrak{su}(n{+}1), \quad \mathfrak{Isom}(\mathbb{H}\mathbb{P}^n) \cong \mathfrak{sp}(n{+}1)$$

- Cayley plane: $\mathfrak{Isom}(\mathbb{O}\mathbb{P}^2) \cong \mathfrak{f}_4$

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- Rosenfeld $\mathbb{OP}^2 \cong (\mathbb{R} \otimes \mathbb{O})\mathbb{P}^2 \to$ projective planes over $\mathbb{A}_L \otimes \mathbb{A}_R$:

  $$(\mathbb{R}\otimes\mathbb{O})\mathbb{P}^2 \to \mathfrak{f}_4, \quad (\mathbb{C}\otimes\mathbb{O})\mathbb{P}^2 \to \mathfrak{e}_6, \quad (\mathbb{H}\otimes\mathbb{O})\mathbb{P}^2 \to \mathfrak{e}_7, \quad (\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2 \to \mathfrak{e}_8$$

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$\mathbb{A}_L \otimes \mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
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$\mathbb{H}$	$\mathfrak{sp}(4,2)$	$\mathfrak{su}(4,2)$	$\mathfrak{so}(8,4)$	$\mathfrak{e}_{7(-5)}$
$\mathbb{O}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(8)}$

Table : Freudenthal magic square [Freudenthal:1954,Rosenfeld:1956,Tits:1966]

$D = 3, \mathcal{N} = 8$ Yang-Mills

- $D = 3$, $\mathcal{N} = 8$ super Yang-Mills field content:

$$\{A_\mu, \quad \phi_i, \quad \lambda_a\}, \qquad i = 1, \ldots 7, \; a = 1, \ldots 8$$

each valued in the adjoint of the gauge group G

- The Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{2}D_\mu\phi^A_i D^\mu\phi^A_i + i\bar{\lambda}^A_a\gamma^\mu D_\mu\lambda^A_a \\ &-\frac{1}{4}g^2 f_{BC}{}^A f_{DE}{}^A \phi^B_i\phi^D_i\phi^C_j\phi^E_j - g f_{BC}{}^A\phi^B_i\bar{\lambda}^{Aa}\Gamma^i_{ab}\lambda^{Cb}\end{aligned}$$

$D = 3, \mathcal{N} = 8$ Yang-Mills

- $D = 3$, $\mathcal{N} = 8$ super Yang-Mills field content:

$$\{A_\mu, \quad \phi_i, \quad \lambda_a\}, \qquad i = 1, \dots 7, \; a = 1, \dots 8$$

  each valued in the adjoint of the gauge group G

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- Γ can be represented by the octonionic structure constants:

$$\Gamma^i_{ab} = i(\delta_{bi}\delta_{a0} - \delta_{b0}\delta_{ai} + C_{iab})$$

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ description of $D = 3, \mathcal{N} = 1, 2, 4, 8$ Yang-Mills

- $D = 3$ super Yang-Mills field content:

$$\{A_\mu, \quad \phi_i, \quad \lambda_a\}, \qquad i = 1, \ldots \mathcal{N} - 1, \quad a = 1, \ldots \mathcal{N}$$

- $\mathbb{A}$-valued fields:

$$A_\mu \in \mathrm{Re}\mathbb{A} \qquad \phi \in \mathrm{Im}\mathbb{A} \qquad \lambda \in \mathbb{A}$$

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Unified description for $\mathcal{N} = 1, 2, 4, 8$:

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{1}{2}\langle D_\mu\phi, D^\mu\phi\rangle + i\bar{\lambda}\not{D}\lambda - \frac{1}{4}g^2\langle[\phi,\phi],[\phi,\phi]\rangle + \frac{i}{2}g[\bar{\lambda},\phi,\lambda]$$

- $\lambda \to$ algebra of octonions defined over the Grassmanns
- $[a, b, c] := (ab)c - a(bc)$ is an alternating function: crucial for susy

Squaring $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ Yang-Mills

Tensor left and right division super Yang-Mills multiplets:

$$\{A_\mu \in \mathrm{Re}\mathbb{A}_L, \quad \phi \in \mathrm{Im}\mathbb{A}_L, \quad \lambda \in \mathbb{A}_L\}$$

$$\otimes$$

$$\{\tilde{A}_\mu \in \mathrm{Re}\mathbb{A}_R, \quad \tilde{\phi} \in \mathrm{Im}\mathbb{A}_R, \quad \tilde{\lambda} \in \mathbb{A}_R\}$$

Supergravity theory valued in both $\mathbb{A}_L$ and $\mathbb{A}_R$:

$$\boxed{g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}, \quad \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}}$$

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In the maximal case of $\mathbb{A}_L, \mathbb{A}_R = \mathbb{O}$:

$$\Psi \in \mathbf{16} \qquad \varphi \in \mathbf{128} \qquad \chi \in \mathbf{128}' \quad \text{of} \quad \mathfrak{so}(16)$$

H algebra

Maximal compact subgroup of U-duality:

Right global bosonic symmetries

$$\mathfrak{h}(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + \mathbb{A}_L \otimes \mathbb{A}_R,$$

Left global bosonic symmetries

Product of Left/Right global fermionic symmetries

$$Q \in \mathbb{A} \quad \Rightarrow \quad Q \otimes \tilde{Q} \in \mathbb{A}_L \otimes \mathbb{A}_R$$

U-duality

- The U-dualities G are realised non-linearly on the scalars in G/H (See Mario Trigiante's lectures)

- U-duality Lie algebra is given by the triality construction

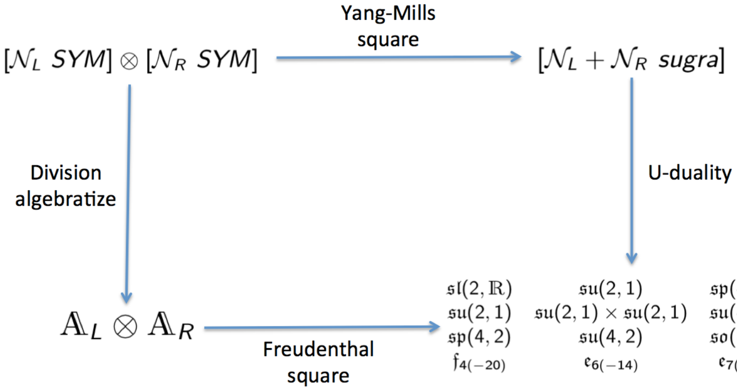
$$\underbrace{\mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + (\mathbb{A}_L \otimes \mathbb{A}_R)}_{\mathfrak{h}(\mathbb{A}_L, \mathbb{A}_R)} + \underbrace{(\mathbb{A}_L \otimes \mathbb{A}_R)^2}_{\text{"scalars"}},$$

- $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded Lie algebra structure:

$$\begin{aligned} [(a \otimes b, 0, 0), (0, a' \otimes b', 0)] &= \quad (0, 0, \overline{aa'} \otimes \overline{bb'}), \\ [(0, 0, a \otimes b), (a' \otimes b', 0, 0)] &= \quad (0, \overline{aa'} \otimes \overline{bb'}, 0), \\ [(0, a \otimes b, 0), (0, 0, a' \otimes b')] &= -(\overline{aa'} \otimes \overline{bb'}, 0, 0). \end{aligned}$$

$\rightarrow$ Freudenthal magic square! [LB-Duff-Hughes-Nagy:2013]

$D = 3$ magic square of supergravities



$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{su}(2,1)$	$\mathfrak{sp}(4,2)$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{su}(2,1)$	$\mathfrak{su}(2,1)\times\mathfrak{su}(2,1)$	$\mathfrak{su}(4,2)$	$\mathfrak{e}_{6(-14)}$
$\mathfrak{sp}(4,2)$	$\mathfrak{su}(4,2)$	$\mathfrak{so}(8,4)$	$\mathfrak{e}_{7(-5)}$
$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(8)}$

$D = 3, 4, 6, 10$ super-Yang-Mills

Division algebras $\Leftrightarrow$ Lorentzian spacetime

$$\mathfrak{sl}(2, \mathbb{A}) \cong \mathfrak{so}(1, 2 + \dim \mathbb{A})$$

[Gunaydin-Gursey:1974, Kugo-Townsend:1982, Sudbery:1984 . . .]

$D = 3, 4, 6, 10$ super-Yang-Mills

$$\text{Division algebras} \quad \Leftrightarrow \quad \text{Lorentzian spacetime}$$

$$\mathfrak{sl}(2, \mathbb{A}) \quad \cong \quad \mathfrak{so}(1, 2 + \dim \mathbb{A})$$

[Gunaydin-Gursey:1974, Kugo-Townsend:1982, Sudbery:1984 ...]

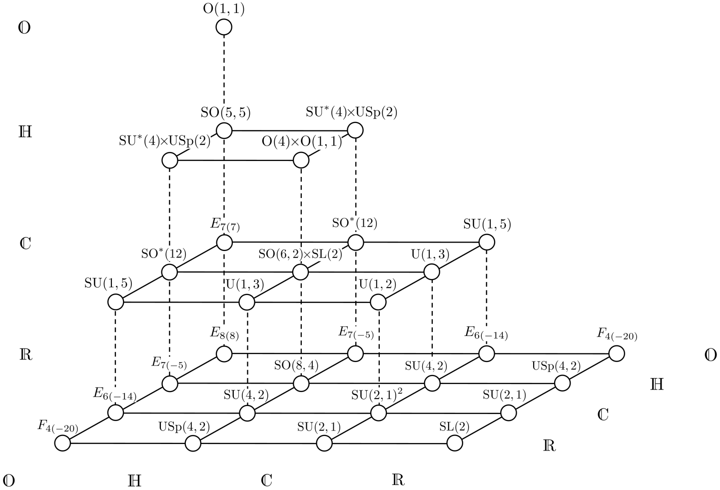
- $\mathcal{N} = 1$ super Yang-Mills in $D = 2 + \dim \mathbb{A} = 3, 4, 6, 10$:

$$\mathcal{L}(\mathbb{A}_n) = -\frac{1}{4}\langle F, F\rangle + \frac{1}{2}\langle \lambda^\dagger, D\lambda\rangle, \qquad \dim \mathbb{A}_n = n$$

$$\boxed{(D = n+2, \mathcal{N}) \text{ super Yang-Mills theory} \Leftrightarrow \text{an ordered pair } \mathbb{A}_n \subseteq \mathbb{A}_{n\mathcal{N}}}$$

[Anastasiou-LB-Duff-Hughes-Nagy:2013, 2014]

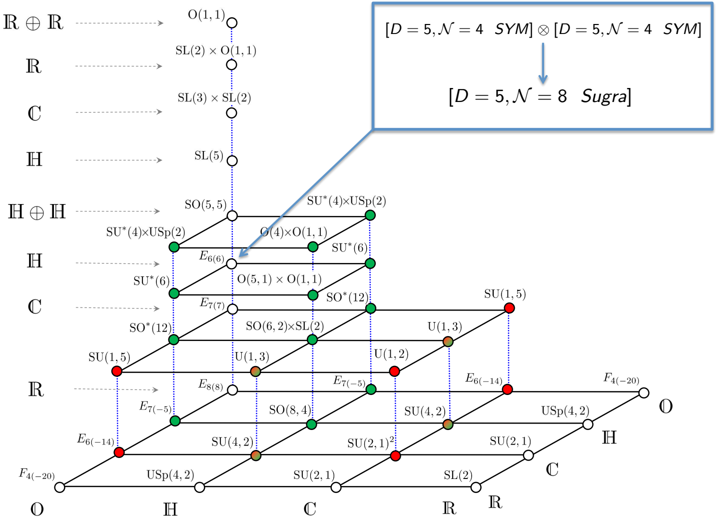
$D = 3, 4, 6, 10$ Magic Pyramid



Magic pyramid algebra: $\mathfrak{MPyr}(\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}_L}, \mathbb{A}_{n\mathcal{N}_R})$

[Anastasiou-LB-Duff-Hughes-Nagy:2013]

$3 \leq D \leq 10$ Pyramid



Generalised pyramid algebra: $\mathfrak{Pyr}(\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R])$ [Anastasiou-LB-Hughes-Nagy:2015]

$D = 5, \mathcal{N} = 4$ super Yang-Mills at strong coupling

M5-branes $\rightarrow$ $D = 6, \mathcal{N} = (2,0)$ superconformal theory

$$\{B_{\mu\nu}, \phi^{[ab]}, \lambda^a\}_{tensor} \qquad \mathfrak{so}(2,6) \oplus \mathfrak{usp}(4) \subset \mathfrak{osp}(8^\star|4)$$

$$\begin{array}{cccc} & B_{\mu\nu} & \phi^{[ab]} & \lambda^a \\ g_{YM} \to \infty & \big\uparrow & \big\uparrow & \big\uparrow \\ & A_\mu & \phi^{[ab]} & \lambda^a \end{array}$$

D4-branes $\rightarrow$ $D = 5, \mathcal{N} = 4$ super Yang-Mills theory

$$\{A_\mu, \phi^{[ab]}, \lambda^a\}_{sym} \qquad \text{R-symmetry} \quad \mathfrak{usp}(4)$$

[Nahm:1979, Blencowe-Duff:87, Witten:95, Strominger:95, Townsend:95, Maldacena:97...]

(2,0) squared?

$$(D = 6, \mathcal{N} = (2,0) \;\; \textit{tensor})^2$$

↑ M-theory uplift

$$D = 5, \mathcal{N} = 8 \;\; \text{sugra} \xleftarrow{\text{convolutive square}} (D = 5, \mathcal{N} = 4 \;\; \text{SYM})^2$$

(2,0) squared?

Use off-shell dictionary prescription:

- $\mathcal{N} = (4,0)$ superconformal $\mathfrak{osp}(8^\star|8)$ multiplet [Nahm:1978]:

  $$(\mathbf{5},\mathbf{1};\mathbf{1}) + (\mathbf{4},\mathbf{1};\mathbf{8}) + (\mathbf{3},\mathbf{1};\mathbf{27}) + (\mathbf{2},\mathbf{1};\mathbf{48}) + (\mathbf{1},\mathbf{1};\mathbf{42})$$

  of $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{usp}(8)$

- U-duality: $\phi \in E_{6(6)}/\,\mathrm{USp}(8)$

- $B_{\mu\nu}^{[AB]}$ in linear **27** of $E_{6(6)}$

- $B_{\mu\nu} \otimes \tilde{B}_{\rho\sigma}, \quad B_{\mu\nu} \otimes \tilde{\lambda}^a \ldots \rightarrow$ exotic $D = 6$ covariant fields:

$$\boxed{C_{\mu\nu\rho\sigma}, \quad \psi^A_{\mu\nu}, \quad B^{[AB]}_{\mu\nu}, \quad \lambda^{[ABC]}, \quad \phi^{[ABCD]}}$$

Amplitude relations: [Huang-Lipstein:2010; Czech-Huang-Rozali:2012; Chiodaroli-Gunaydin-Roiban:2012; Carrasco-Chiodaroli-Gunaydin-Roiban:2013]

(2,0) squared?

Unconventional "dual graviton" $C_{\mu\nu\rho\sigma} = B_{\mu\nu} \otimes \tilde{B}_{\rho\sigma}|_{105}$:

$$C_{\mu\nu\rho\sigma} = C_{[\mu\nu][\rho\sigma]} = C_{[\rho\sigma][\mu\nu]}, \qquad C_{[\mu\nu\rho]\sigma} = 0$$

- Generalised gauge symmetries:

$$\delta B_{\mu\nu} = \partial_{[\mu}\lambda_{\nu]} \qquad \rightarrow \qquad \delta C_{\mu\nu\rho\sigma} = \partial_{[\mu}\xi_{\nu]\rho\sigma} + \partial_{[\rho}\xi_{\sigma]\mu\nu} - 2\partial_{[\mu}\xi_{\nu]\rho\sigma}$$

- Gauge for gauge follows automatically from dictionary $\delta\lambda_\nu = \lambda$
- Gauge invariant field strengths:

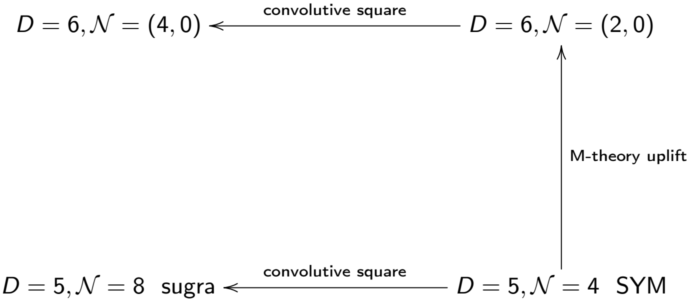
$$H_{\mu\nu\rho} \otimes \tilde{H}_{\mu\nu\rho} \qquad \rightarrow \qquad G_{\mu\nu\rho\sigma\tau\lambda} := 9\partial_{[\mu} C_{\nu\rho][\sigma\tau,\lambda]} = G_{[\sigma\tau\lambda][\mu\nu\rho]}$$

- Equations of motion and Bianchi identities:

$$dH = 0, \quad d\tilde{H} = 0 \qquad \rightarrow \qquad G_{[\mu\nu\rho\sigma]\tau\lambda} = 0, \quad \partial_{[\kappa} G_{\mu\nu\rho]\sigma\tau\lambda} = 0$$

$$H = \star H, \quad \tilde{H} = \star\tilde{H} \qquad \rightarrow \qquad G = \star G = G\star$$

(2,0) squared?


$D = 6, \mathcal{N} = (4,0)$
convolutive square
$D = 6, \mathcal{N} = (2,0)$
M-theory uplift
$D = 5, \mathcal{N} = 8$ sugra
convolutive square
$D = 5, \mathcal{N} = 4$ SYM

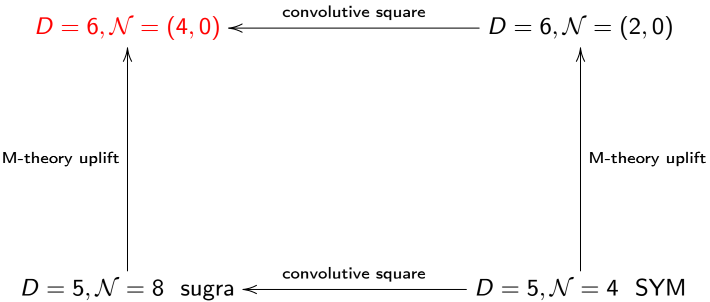
(4,0) superconformal limit of M-theory on T^6

"Squaring" reproduces Chris Hull's proposal for an exotic superconformal $D = 6, \mathcal{N} = (4,0)$ theory [Hull:2000, Hull:2001]:

$$C_{\mu\nu\rho\sigma}, \quad \psi^{A}_{\mu\nu}, \quad B^{[AB]}_{\mu\nu}, \quad \lambda^{[ABC]}, \quad \phi^{[ABCD]}$$

- Strong gravitational coupling limit of $D = 5, \mathcal{N} = 8$ supergravity
- Requires: $E_{6(6)}$ U-duality, 42 scalars, 27 2-forms plus conformal symmetry
- The unique $D = 6, \mathcal{N} = 8$ supergravity theory fails on all counts
- Hull showed how the free $D = 6, \mathcal{N} = (4,0)$ theory dimensionally reduces to conventional $D = 5, \mathcal{N} = 8$ supergravity
- $C_{\mu\nu\rho\sigma} \to h_{\mu\nu}, \tilde{h}_{\mu\nu}, \bar{h}_{\mu\nu}$ all identified by $D = 6$ two-sided self-duality

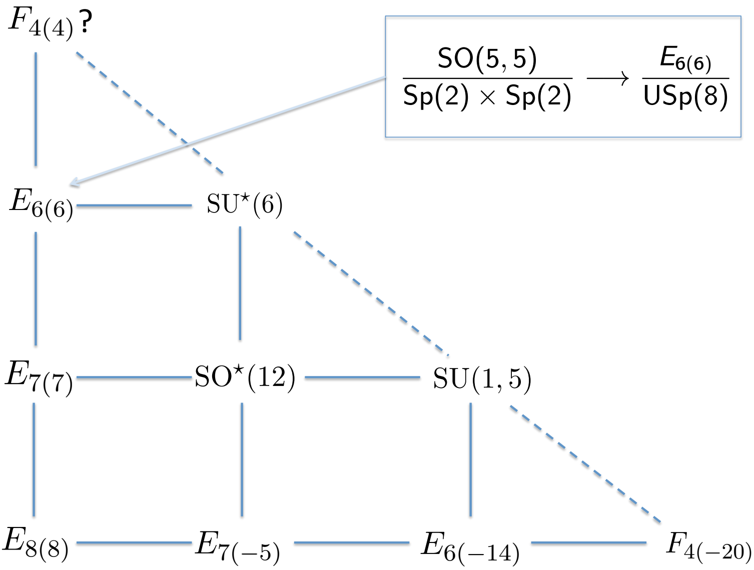
All roads lead to (4, 0)



Conjecture: the exotic (4, 0) theory is the strong coupling limit of $D = 5, \mathcal{N} = 8$ supergravity [Hull:2000, Hull:2001]

The most symmetric phase of M-theory

Conformal Pyramid



Component form

Wess-Zumino gauge $\varphi_\mu|_{WZ} = V|_{WZ} \otimes \tilde{A}_\nu$:

$$\begin{aligned} g_{\mu\nu} + B_{\mu\nu} &= A_\mu & \otimes \quad & \tilde{A}_\nu & & \delta Z_{\mu\nu} &= \nabla_\nu \alpha_\mu(L) + \nabla_\mu \alpha_\nu(R) \\ \psi_\nu &= \chi & \otimes \quad & \tilde{A}_\nu & \Rightarrow \quad & \delta\psi_\mu &= \nabla_\mu \eta \\ V_\nu &= D & \otimes \quad & \tilde{A}_\nu & & \delta V_\mu &= \nabla_\mu \Lambda \end{aligned}$$

Component form

Wess-Zumino gauge $\varphi_\mu|_{WZ} = V|_{WZ} \otimes \tilde{A}_\nu$:

$$\begin{array}{llll} g_{\mu\nu} + B_{\mu\nu} & = A_\mu & \otimes & \tilde{A}_\nu \\ \psi_\nu & = \chi & \otimes & \tilde{A}_\nu \\ V_\nu & = D & \otimes & \tilde{A}_\nu \end{array} \Rightarrow \begin{array}{ll} \delta Z_{\mu\nu} & = \nabla_\nu \alpha_\mu(L) + \nabla_\mu \alpha_\nu(R) \\ \delta\psi_\mu & = \nabla_\mu \eta \\ \delta V_\mu & = \nabla_\mu \Lambda \end{array}$$

Consistency check:

$$\delta_\epsilon^{WZ} V = \delta_\epsilon V + \Lambda_{WZ} + \bar{\Lambda}_{WZ}$$

$$\delta_\epsilon^{WZ} \varphi_\nu = \delta_\epsilon \varphi_\nu + S_\nu^{WZ} + \bar{S}_\nu^{WZ}$$

$\rightarrow$ consistently preserves the WZ gauge choice imposed on φ_ν

Equations of motion

Off-shell Lorentz multiplet:

$$\mathcal{V}^{ab}|_{WZ} \sim (\Omega_{\mu ab}{}^{-}, \psi_{ab}, -2V_{ab}{}^{+})$$

[deRoo et al:1991]

Transformation rules

$$\delta\mathcal{V}^{ab} = \Lambda^{ab} + \bar{\Lambda}^{ab} + \delta_{(a,\lambda,\epsilon)}\mathcal{V}^{ab}.$$

follow from dictionary:

$$\mathcal{V}^{ab} = V \star \Phi \star F^{ab}$$

$$\Lambda^{ab} = \Lambda \star \Phi \star F^{ab}$$

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$$\Lambda^{ab} = \Lambda \star \Phi \star F^{ab}$$

The corresponding Riemann tensors including torsion terms are given by

$$R^{+}_{\mu\nu\rho\sigma} = -F_{\mu\nu} \star \Phi \star \tilde{F}_{\rho\sigma} = R^{-}_{\rho\sigma\mu\nu}$$

Yang-Mills equations ⇔ Einstein's equations