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Semiclassical calculation of (n-point) spectral correlation functions for chaotic systems

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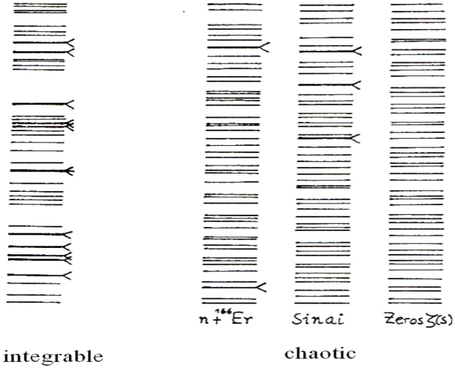
Universal spectral statistics

classical chaos $\Rightarrow$ universal statistics of **quantum** energy levels,
in agreement with random matrix theory
(BGS conjecture)



Correlation functions

- **level density** $\rho(E) = \sum_i \delta(E - E_i)$
- ***n*-point correlation function**

$$R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) = \langle \rho(E + \epsilon_1)\rho(E + \epsilon_2) \ldots \rho(E + \epsilon_n)\rangle_E$$

(we take $\bar{\rho} = 1$)

- for chaotic systems **without time reversal invariance:** agreement with prediction from **Gaussian Unitary Ensemble** (average over hermitian matrices)

$$R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) \;=\; \det\left[\frac{\sin(\pi(\epsilon_j - \epsilon_k))}{\pi(\epsilon_j - \epsilon_k)}\right]_{j,k}$$

Correlation functions

- for chaotic systems **with time reversal invariance:** agreement with prediction from **Gaussian Orthogonal Ensemble** (average over real symmetric matrices)

$$R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) = \mathrm{Pf}\begin{pmatrix} D(\epsilon_i - \epsilon_j) & S(\epsilon_i - \epsilon_j) \\ -S(\epsilon_i - \epsilon_j) & I(\epsilon_i - \epsilon_j) \end{pmatrix}$$

$$S(x) = \frac{\sin(\pi x)}{\pi x}$$

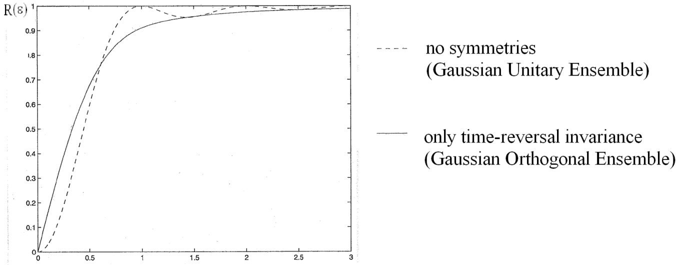
$$D(x) = \int_0^1 du\, u \sin(\pi u x)$$

$$I(x) = -\int_1^\infty \frac{du}{u} \sin(\pi u x).$$

2-point function

$$
\begin{aligned}
R_2(\epsilon_1, \epsilon_2) &= \langle \rho\,(E+\epsilon_1)\,\rho\,(E+\epsilon_2)\rangle \\
&= \mathrm{Re}\left(\sum_m c_m \left(\frac{1}{\epsilon_1-\epsilon_2}\right)^m + \sum_m d_m \left(\frac{1}{\epsilon_1-\epsilon_2}\right)^m e^{2\pi i(\epsilon_1-\epsilon_2)}\right)
\end{aligned}
$$

c_m, d_m predicted by random matrix theory, depend only on symmetry



2-point function

use **Gutzwiller's trace formula**

$$\rho(E) \approx \bar{\rho} + \frac{1}{\pi\hbar}\,\mathrm{Re}\sum_{\text{per. orbits } p} T_p^{\mathrm{prim}} F_p\, e^{iS_p(E)/\hbar}$$

$$\bar{\rho} = 1 \quad \text{(for convenience)} \qquad T_p^{\mathrm{prim}} = \text{primitive period}$$

$$F_p = \frac{1}{\sqrt{|\det(M_p - I)|}}\, e^{-i\mu_p \frac{\pi}{2}} \qquad S_p = \text{action}$$

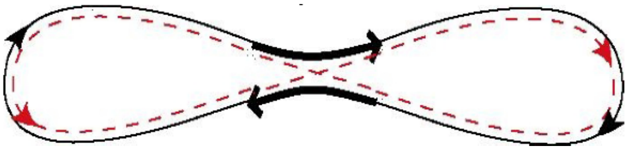
$$R_2(\epsilon_1, \epsilon_2) \approx 1 + \frac{1}{(\pi\hbar)^2}\mathrm{Re}\sum_{p,q}\left\langle T_p^{\mathrm{prim}} F_p T_q^{\mathrm{prim}} F_q^* e^{i(S_p(E+\epsilon_1) - S_q(E+\epsilon_2))/\hbar}\right\rangle_E$$

$\Rightarrow$ need pairs of orbits with small action difference

action correlations (Argaman et al 1993)

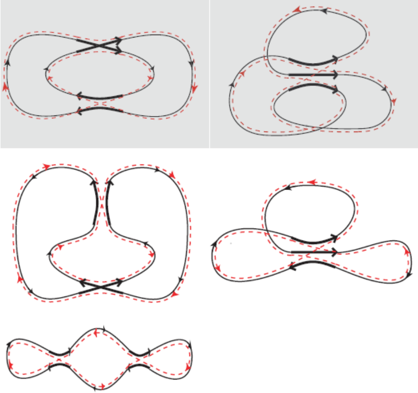
2-point function

- Diagonal approximation:
  $q = p$ or time reversed of p
  $\Rightarrow \frac{1}{(\epsilon_1 - \epsilon_2)^2}$ term
  (Hannay & Ozorio de Almeida, Berry)
  sum rule: $\sum_p T_p^2 |F_p|^2 \delta(T_p - T) \approx T$
- Sieber-Richter pairs



$\Rightarrow \frac{1}{(\epsilon_1 - \epsilon_2)^3}$ term for time rev. inv. systems

2-point function

- 

  $\Rightarrow \frac{1}{(\epsilon_1 - \epsilon_2)^4}$ term
- etc ...
- for oscillatory terms: need improved semiclassical approximation (Riemann-Siegel lookalike formula, Berry & Keating)

Agreement with random matrix theory ☺

S.M., Heusler, Braun, Haake, Altland, 2004 & 2006; Heusler, S.M., Altland Braun Haake 2007 & 2009, Keating & S.M. 2007

n-point functions

use

$$\rho(E) \approx \bar{\rho} + \frac{1}{\pi\hbar} \,\mathrm{Re} \sum_{\text{per. orbits } p} T_p^{\mathrm{prim}} F_p \, e^{iS_p(E)/\hbar}$$

for factors in

$$R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) = \langle \rho(E+\epsilon_1)\rho(E+\epsilon_2)\ldots\rho(E+\epsilon_n)\rangle_E$$

two kinds of orbits:

- p-orbits contribute with $e^{iS_p(E+\epsilon_j)/\hbar}$
- q-orbits contribute with $e^{-iS_q(E+\eta_k)/\hbar}$

(after relabeling energy increments)
need small action difference

$$\Delta S = \sum_{j=1}^{J} S_{p_j} - \sum_{k=1}^{K} S_{q_k}$$

n-point functions

use

$$\rho(E) \approx \bar{\rho} + \frac{1}{\pi\hbar} \operatorname{Re} \sum_{\text{per. orbits } p} T_p^{\text{prim}} F_p \, e^{iS_p(E)/\hbar}$$

for factors in

$$R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) = \langle \rho(E+\epsilon_1)\rho(E+\epsilon_2)\ldots\rho(E+\epsilon_n)\rangle_E$$

further book-keeping:

- Taylor expand action using $\frac{dS}{dE} = T$
- get period factors using derivatives of actions
- contributions with Weyl term related to lower-order correlations

Contributing orbits

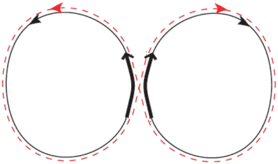
contributions with small small action difference

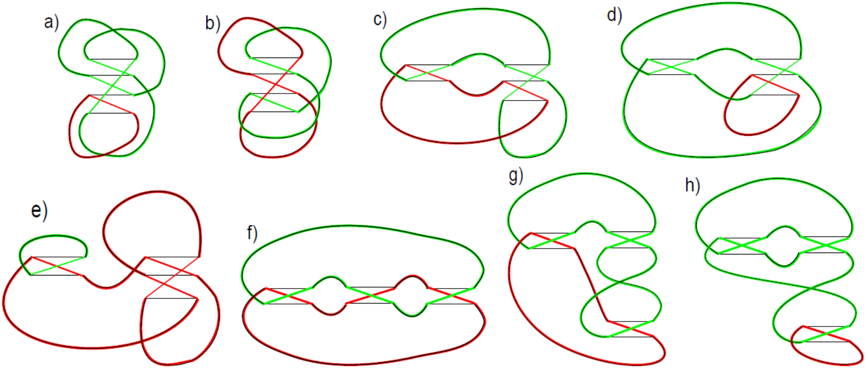
$$\Delta S = \sum_{j=1}^{J} S_{p_j} - \sum_{k=1}^{K} S_{q_k}$$

- **diagonal approximation:** *p*- and *q*-orbits coincide pairwise
- *p*- and *q*-orbits coincide up to connections in **encounters**
- can also have mix of both mechanisms

Contributing orbits

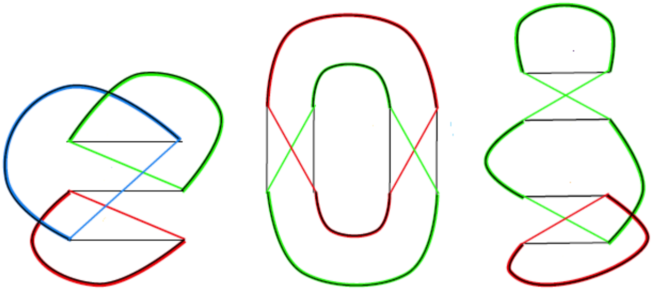
3-point function: reconnections in one orbit give 2 orbits





Contributing orbits

Some contributions to 4-point function:



Semiclassical calculation

- action difference: e.g. for 2-encounter product of stable and unstable deviations between encounter stretches
  (Turek & Richter 2003, Spehner 2003)
- ergodicity: Hannay-Ozorio de Almeida sum rule, probability for encounters
- each link gives $(\epsilon_j - \eta_k)^{-1}$ (j = index of p-orbit, k = index of q-orbit)
- encounter contributions cancel some link contributions

Result proportional to:

$$\prod_j \frac{\partial}{\partial \epsilon_j} \prod_j \frac{\partial}{\partial \eta_k} \sum_{\text{diagrams}} (-1)^{\#\,\text{enc}} \prod_{\text{links (uncancelled)}} (\epsilon_j - \eta_k)^{-1}$$

Results

- general diagrammatic rule for non-oscillatory contributions
- for systems with and without time-reversal invariance:
  agreement with RMT up to 5-point correlation function for leading few orders,
- for systems without time-reversal invariance:
  proof that encounter contributions cancel in all orders, for abitrary $n-$point functions
  based on mapping to matrix model
  (diagonal approximation evaluated in Nagao & S.M. 2009)

Matrix model

Encounter contributions proportional to

$$\prod_{j=1}^{J} \frac{\partial}{\partial \epsilon_j} \prod_{k=1}^{K} \frac{\partial}{\partial \eta_k} \left[r^{J+K} \right] \int d\mu(Z) \exp\left(-\sum_{q \geq 2} \mathrm{Tr}[X(ZZ^\dagger)^q - (Z^\dagger Z)^q Y] \right)$$

where

$$d\mu(Z) = \exp\left(-\mathrm{Tr}[XZZ^\dagger - Z^\dagger ZY] \right) dZ$$

$$X \propto \mathrm{diag}(\underbrace{\epsilon_1, \ldots, \epsilon_1}_{r \text{ copies}}, \epsilon_2, \ldots, \epsilon_2, \ldots)$$

$$Y \propto \mathrm{diag}(\underbrace{\eta_1, \ldots, \eta_1}_{r \text{ copies}}, \eta_2, \ldots, \eta_2, \ldots)$$

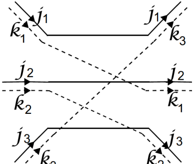
Matrix model: Motivation

$$\prod_{j=1}^{J} \frac{\partial}{\partial \epsilon_j} \prod_{k=1}^{K} \frac{\partial}{\partial \eta_k} \left[r^{J+K} \right] \int d\mu(Z) \exp \left(- \sum_{q \geq 2} \mathrm{Tr}[X(ZZ^\dagger)^q - (Z^\dagger Z)^q Y] \right)$$

- expansion of exponential and Wick's theorem lead to terms like

$$\int d\mu(Z) \mathrm{Tr}[X(ZZ^\dagger ZZ^\dagger)] \mathrm{Tr}[X(ZZ^\dagger ZZ^\dagger)]$$

- contraction lines analogous to links, give factors $(\epsilon_j - \eta_k)^{-1}$
- traces analogous to encounters, with Z_{jk} and Z^*_{jk} corresponding to 'ports' at the ends of encounter stretches and j, k corresponding to orbits:



Matrix model: Evaluation

- can do integral exactly:

$$\frac{\det(e^{X_j - Y_k}\mathrm{Ei}(2N, X_j - Y_k))}{\det((X_j - Y_k)^{-1})}$$

- all terms in result vanish either due to $[r^{J+K}]$ or due to derivatives
- off-diagonal contributions to all correlation functions cancel (for time-reversal invariant systems)

Conclusions

- n-point correlations of chaotic systems determined by multiple sums over orbits
- contributions arise if orbits are identical (up to time reversal) or differ in encounters
- n-point correlation functions agree with RMT
- with time-reversal invariance: checked leading few orders up to $n = 5$ (for non-oscillatory terms)
- without time-reversal invariance: cancellation of off-diagonal contributions shown using matrix integral

Oscillatory terms

Need improved semiclassical approximation:
Riemann-Siegel lookalike formula (Berry, Keating 1990)

$$\rho(E) = -\frac{1}{2\pi}\mathrm{Im}\frac{\partial}{\partial E'}\frac{det(E-H)}{\det(E'-H)}\Big|_{E'=E}$$

$$\det(E-H) = \boxed{e^{-i\pi E}} \times \boxed{\sum_A F_A e^{iS_A(E)/\hbar}} + \text{c.c.}$$

sum over sets of classical periodic orbits shorter than $T_H/2$

Derivation:

- Gutzwiller formula for $\mathrm{tr}\frac{1}{E-H}$
- $\det(E-H) = \exp\mathrm{tr}\ln(E-H) = \exp\left(\int \mathrm{tr}\frac{1}{E-H}\right)$
- expand exponential
- get relation between short and long orbits from