SU(2)$_k$ WZW model solutions in string field theory

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Workshop on Fundamental Aspects of String Theory

Introduction

- We will discuss open string field theory involving the $SU(2)_k$ WZW model
  - A test of OSFT on more complicated background
- OSFT solutions are conjectured to describe boundary states
  - We observe transitions between boundary states
  - There seem to be interesting „selection rules“ regarding conventional (Cardy) boundary states
  - They could lead to better understanding of boundary RG flow
- We can search for non-conventional boundary states
- Earlier work by Michishita (hep-th/0105246)

- We work with the traditional bosonic open string field theory with the action

$$S = -\frac{1}{g_o^2}\int\left(\frac{1}{2}\Psi * Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi\right)$$

- We use the level truncation approach
  - Numerical approach
  - We impose Siegel gauge

- The theory is more complicated than free boson or minimal models
  - We cannot reach very high levels
  - Lower precision of results
  - Still good enough to identify most solutions

SU(2)$_k$ WZW model

- SU(2) group elements can be parameterized using 3 angles as

$$g = \begin{pmatrix} \cos\theta + i\cos\psi\sin\theta & ie^{i\phi}\sin\theta\sin\psi \\ ie^{-i\phi}\sin\theta\sin\psi & \cos\theta - i\cos\psi\sin\theta \end{pmatrix}$$

- They are generated by 3 operators: $J^{\pm}$, J^3
- In SU(2)$_k$ WZW model, the operators are lifted to currents, which have mode algebra

$$\begin{aligned} [J^{\pm}_m, J^{\pm}_n] &= 0, \\ [J^3_m, J^3_n] &= \frac{mk}{2}\delta_{m+n,0}, \\ [J^{\pm}_m, J^{\mp}_n] &= \pm 2J^3_{m+n} + mk\delta_{m+n,0}, \\ [J^3_m, J^{\pm}_n] &= \pm J^{\pm}_{m+n}. \end{aligned}$$

- Primary fields are have a structure following irreducible SU(2) representations
  - We label them as $|j, m\rangle$
  - The range of *j* is restricted by the level *k* to $j = 0, \ldots, \frac{k}{2}$
  - *m* has the usual range $m = -j, \ldots, j$
  - The currents act on primary fields as

$$\begin{aligned} J_n^a|j,m\rangle &= 0, \quad n > 0, \\ J_0^3|j,m\rangle &= m|j,m\rangle, \\ J_0^+|j,m\rangle &= \alpha_{j,m}^+|j,m+1\rangle, \\ J_0^-|j,m\rangle &= \alpha_{j,m}^-|j,m-1\rangle, \end{aligned}$$

  - Hilbert space is spanned by states

$$J_{-n_1}^{a_1} \ldots J_{-n_l}^{a_l}|j,m\rangle$$

Boundary states

We distinguish two types of boundary states

- Boundary states which preserve half of the bulk symmetry
  - They satisfy gluing conditions

$$(J_n^a + \Omega^a{}_b(g)\bar{J}^b_{-n})\|B\rangle\rangle = 0$$

  - They are labeled by SU(2) group elements *g* and half-integer *J*
  - For a given *g*, we find the usual Cardy solution, which is given in terms of S-matrix

$$\|J, g\rangle\rangle = \sum_j \frac{S_J{}^j}{\sqrt{S_0{}^j}}|j, g\rangle\rangle = \sum_j B_J^j|j, g\rangle\rangle$$

- Symmetry-breaking boundary states
  - They genericly satisfy only the Virasoro gluing conditions
  - Most of them are not understood

- In OSFT, we impose the following condition

$$J_0^3|\Psi\rangle = 0$$

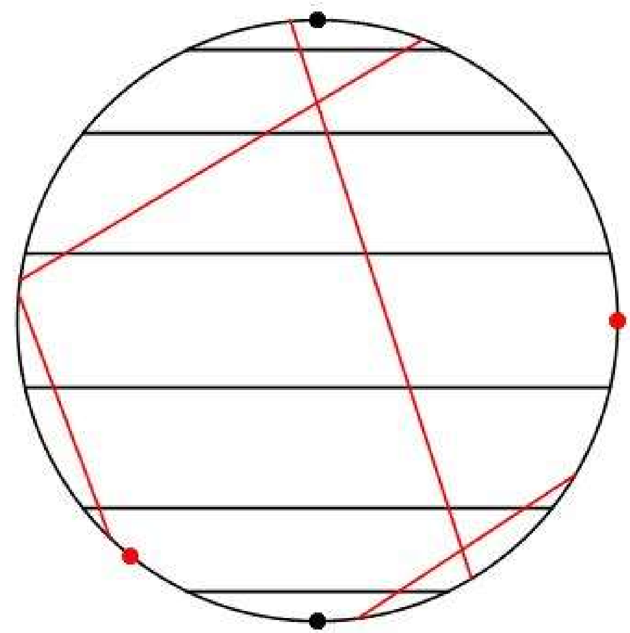
  - Used for fixing SU(2) symmetry of solutions $|\Psi\rangle \to e^{i\lambda_a J_0^a}|\Psi\rangle$

- This implies that solutions preserve the J^3 gluing condition
  - Great simplification
  - Only the parameter θ survives
  - Its range is now from -π to π
  - Group elements simplify to

$$g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

  - Irredicible representations also become diagonal

- Cardy boundary states are associated with SU(2) conjugacy classes
  - Conjugacy classes form either points ($J=0$ or $J=k/2$) or 2-spheres on the SU(2) 3-sphere
  - Branes which preserve the J^3 gluing condition can be nicely visualized as points or lines on a circle
  - The angle θ determines rotation of branes
  - (J,θ)-brane is the same as ($k/2$-J,π-θ)-brane



- The figure shows $k=7$ case as example
  - Branes with $\theta=0$ have black color
  - Branes with $\theta\neq0$ have red color

Observables

We consider the following observables:

- The energy derived from OSFT action
- Ellwood invariants
  - Labeled by bulk primary operators

$$E_{j,m} = 2\pi i \langle E[c\bar{c}\phi_{j,m,-m}V^{aux}]|\Psi - \Psi_{TV}\rangle$$

  - They describe boundary states corresponding to solutions
  - The expected values are

$$E^{exp}_{j,m} = (-1)^{j-m} B_J^{\,j} e^{2im\theta}$$

- The first out-of-Siegel equation Δ_S as a consistency check

$$\Delta_S \equiv -\langle 0|c_{-1}c_0 b_2\,|Q\Psi + \Psi * \Psi\rangle$$

Regular solutions

- Solutions describing Cardy boundary states, which preserve half of the bulk symmetry

- In these examples, we consider *k=4* and initial boundary condition *J=1*
- There are three groups solutions which satisfy the reality condition

$$E_{j,m} = (-1)^{2j} E^*_{j,-m}$$

- We have reached level 11
- To reduce the amount of data, we show only extrapolations to infinite level
  - Maximum order extrapolations using polynolials in *1/L*

- First solution
  - Based on energy, it represents a ½-brane
  - The invariant $E_{2,2}$ is real
    $\Rightarrow$ we can determine θ exactly
  - The angle is $\theta=\pi/4$
  - All invariants are consistent with this angle
  - It satisfies Δ_S quite well
  - There are 3 more solutions related by rotations
  - No solution for $\theta=0$

	Energy	$E_{0,0}$	Δ_S		
∞	0.9320	0.919	-0.0010		
σ	0.0003	0.004	0.0001		
Exp.	0.930605	0.930605	0		
	$E_{1/2,1/2}$	$E_{1/2,-1/2}$			
∞	$0.482+0.487i$	$-0.482+0.487i$			
σ	$0.002+0.002i$	$0.002+0.002i$			
Exp.	$0.5+0.5i$	$-0.5+0.5i$			
	$E_{1,1}$	$E_{1,0}$	$E_{1,-1}$		
∞	-0.009	0.09	-0.009		
σ	0.004	0.03	0.004		
Exp.	0	0	0		
	$E_{3/2,3/2}$	$E_{3/2,1/2}$	$E_{3/2,-1/2}$	$E_{3/2,-3/2}$	
∞	$0.482-0.487i$	$0.59+0.54i$	$-0.59+0.54i$	$-0.482-0.487i$	
σ	$0.002+0.002i$	$0.12+0.05i$	$0.12+0.05i$	$0.002+0.002i$	
Exp.	$0.5-0.5i$	$0.5+0.5i$	$-0.5+0.5i$	$-0.5-0.5i$	
	$E_{2,2}$	$E_{2,1}$	$E_{2,0}$	$E_{2,-1}$	$E_{2,-2}$
∞	0.919	$-0.04+0.97i$	0.71	$-0.04-0.97i$	0.919
σ	0.004	$0.08+0.07i$	0.12	$0.08-0.07i$	0.004
Exp.	0.930605	$0.930605i$	0.537285	$-0.930605i$	0.930605

- Second solution
  - Corresponds to a 0-brane
  - Some invariants are real ⇒ it has exactly $\theta=\pi/2$
  - Similar properties as the first solution
  - There is other solution with $\theta=-\pi/2$

	Energy	E_0	Δ_S		
∞	0.537311	0.536	-0.00009		
σ	0.000008	0.001	0.00008		
Exp.	0.537285	0.537285	0		
	$E_{1/2,1/2}$	$E_{1/2,-1/2}$			
∞	$0.704i$	$0.704i$			
σ	$0.001i$	$0.001i$			
Exp.	$0.707107i$	$0.707107i$			
	$E_{1,1}$	$E_{1,0}$	$E_{1,-1}$		
∞	-0.757	-0.761	-0.757		
σ	0.001	0.011	0.001		
Exp.	-0.759836	-0.759836	-0.759836		
	$E_{3/2,3/2}$	$E_{3/2,1/2}$	$E_{3/2,-1/2}$	$E_{3/2,-3/2}$	
∞	$-0.704i$	$-0.69i$	$-0.69i$	$-0.704i$	
σ	$0.001i$	$0.10i$	$0.10i$	$0.001i$	
Exp.	$-0.707107i$	$-0.707107i$	$-0.707107i$	$-0.707107i$	
	$E_{2,2}$	$E_{2,1}$	$E_{2,0}$	$E_{2,-1}$	$E_{2,-2}$
∞	0.536	0.55	0.71	0.55	0.536
σ	0.001	0.02	0.12	0.02	0.001
Exp.	0.537285	0.537285	0.537285	0.537285	0.537285

- Third solution
  - ➢ Slow convergence of invariants
  - ➢ Probably represents also a 0-brane
  - ➢ Only rough agreement of observables
    $\Rightarrow$ there is a small chance that
    it is an exotic solution
  - ➢ It has θ=*0*
  - ➢ There is a second solution with θ=π

	Energy	$E_{0,0}$	Δ_S
∞	0.580	0.528	-0.0110
σ	0.003	0.005	0.0007
Exp.	0.537285	0.537285	0

	$E_{1/2,1/2}$	$E_{1/2,-1/2}$
∞	0.657	-0.657
σ	0.007	0.007
Exp.	0.707107	-0.707107

	$E_{1,1}$	$E_{1,0}$	$E_{1,-1}$
∞	0.689	-0.56	0.689
σ	0.006	0.06	0.006
Exp.	0.759836	-0.759836	0.759836

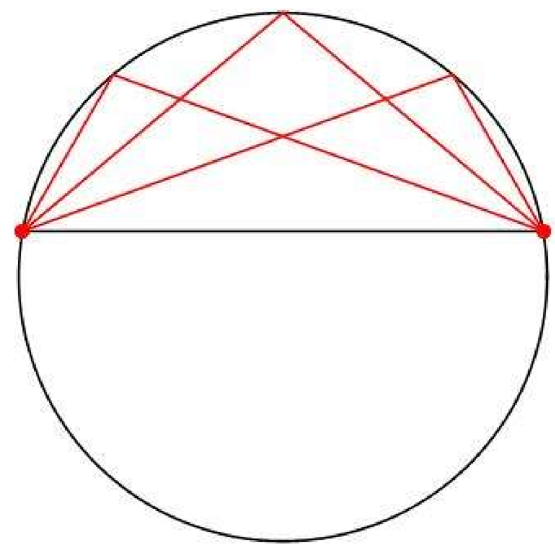
	$E_{3/2,3/2}$	$E_{3/2,1/2}$	$E_{3/2,-1/2}$	$E_{3/2,-3/2}$
∞	0.657	-0.5	0.5	-0.657
σ	0.007	0.1	0.1	0.007
Exp.	0.707107	-0.707107	0.707107	-0.707107

	$E_{2,2}$	$E_{2,1}$	$E_{2,0}$	$E_{2,-1}$	$E_{2,-2}$
∞	0.528	-0.47	0.3	-0.47	0.528
σ	0.005	0.42	0.3	0.42	0.007
Exp.	0.537285	-0.537285	0.537285	-0.537285	0.537285

- Solutions at other k have similar properties
- We can formulate some „selection rules“ regarding θ
  - The best solutions have θ proportional to J_i-J_f

$$\theta = \pm 2|J_i - J_f|\frac{\pi}{k}$$

  - Unless $J=k/2$, branes tend to be on the same half of the circle as the initial brane
  - New branes touch the original one at one point

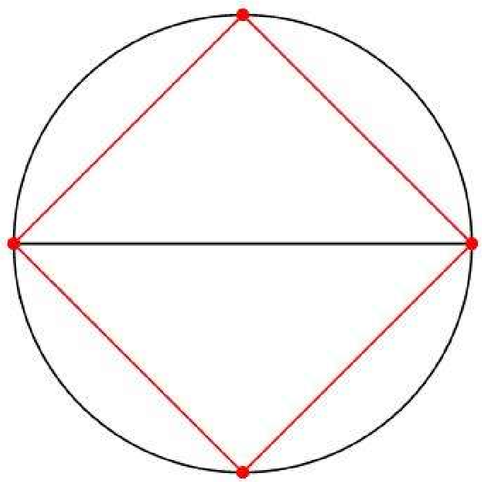


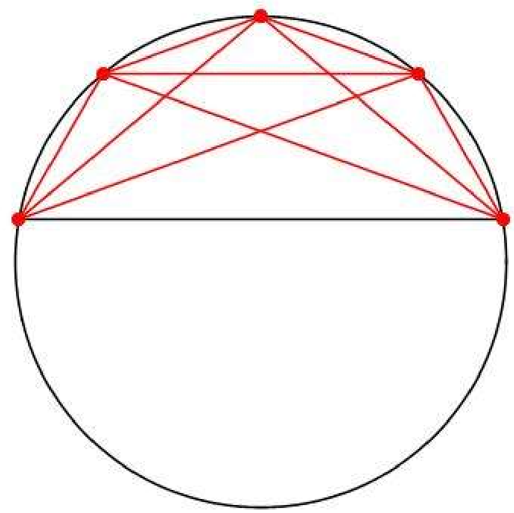
- The example depicts $k=9$ with $J_i=2$

- If we consider also solutions with worse convergence, the rule generalizes to

$$\theta = \pm 2|J_i - J_f + l|\frac{\pi}{k}, \quad l \in \mathbb{Z}$$

- Examples depict *k=4* and *k=9*





SL(2,C) solutions

- SL(2,C) group is complexification of SU(2) ⇒ if we allow complex solutions, we can see solutions describing SL(2,C) gluing conditions
- We generalize the angle θ by adding a new parameter ρ

$$\theta \to \theta - i \log \rho$$

so that

$$e^{in\theta} \to \rho^n e^{in\theta}$$

- Therefore invariants with high m usually have either small or large values
- θ seems to follow the same rule as before

$$\theta = \pm 2|J_i - J_f|\frac{\pi}{k}$$

- ρ seems to be generic

- 0-brane solution at $k=2$ (level 14)
- $\theta=-\pi/k+i\log 2.070$
- Invariants are not symmetric
  ⇒solution does not satisfy reality conditions
- The action is real
  ⇒ pseudo-real solution
- It excites the marginal field, but with imaginary value

	Energy	E_0	Δ_S
∞	0.707093	0.7076	-0.000016
σ	0.000003	0.0001	0.000002
Exp.	0.707107	0.707107	0
	$E_{1/2,1/2}$	$E_{1/2,-1/2}$	
∞	$-1.744i$	$-0.405i$	
σ	$0.015i$	$0.003i$	
Exp.	$-1.74064i$	$-0.406234i$	
	$E_{1,1}$	$E_{1,0}$	$E_{1,-1}$
∞	-3.029	-0.714	-0.163
σ	0.030	0.014	0.005
Exp.	-3.02982	-0.707107	-0.165026

Exotic solutions

- There are some solutions that are clearly not Cardy boundary states
  $\Rightarrow$ we find unknown boundary states
- Symmetry-breaking boundary states
  - They break J^+, J^- gluing conditions
  - But they still preserve J^3 gluing conditions
- Only a small number of well-behaved exotic solutions (compared to free boson)
  - They appear mainly on boundary states with high J
- Sometimes there are more solutions with similar properties
  - Related by marginal deformations?

- The first exotic solution appears at *k*=*3* and *J*=½
  - Complex at levels levels 2,3
  - Real from level 4
  - Highly symmetric
  - Satisfies out-of-Siegel equations

	Energy	E_0	Δ_S	
∞	1.05624	1.054	-0.00014	
σ	–	0.001	–	
Exp.	1.05605	1.05605	0	
	$E_{1/2,1/2}$	$E_{1/2,-1/2}$		
∞	0.006	-0.006		
σ	0.016	0.016		
Exp.	0	0		
	$E_{1,1}$	$E_{1,0}$	$E_{1,-1}$	
∞	-0.006	1.31	-0.006	
σ	0.016	0.03	0.016	
Exp.	0	1.34332	0	
	$E_{3/2,3/2}$	$E_{3/2,1/2}$	$E_{3/2,-1/2}$	$E_{3/2,-3/2}$
∞	-1.054	0.006	-0.006	1.054
σ	0.001	0.016	0.016	0.001
Exp.	1.05605	0	0	1.05605

- There is a similar solution in M(5,6) because $SU(2)_3$=M(5,6)×U(1)
  - We can predict values of observables
  - Corresponding boundary state should be possible to find analytically

- Two exotic solutions at *k=6* with similar properties

	Energy	$E_{0,0}$	Δ_S				
∞	1.07149	1.0713	-0.000036				
σ	0.00002	0.0011	0.000003				
	$E_{1/2,1/2}$	$E_{1/2,-1/2}$					
∞	0	0					
	$E_{1,1}$	$E_{1,0}$	$E_{1,-1}$				
∞	0.0003	1.667	0.0003				
σ	0.0004	0.005	0.0004				
	$E_{3/2,3/2}$	$E_{3/2,1/2}$	$E_{3/2,-1/2}$	$E_{3/2,-3/2}$			
∞	0	0	0	0			
	$E_{2,2}$	$E_{2,1}$	$E_{2,0}$	$E_{2,-1}$	$E_{2,-2}$		
∞	-0.0003	-0.02	1.66	-0.02	-0.0003		
σ	0.0004	0.06	0.19	0.06	0.0004		
	$E_{5/2,5/2}$	$E_{5/2,3/2}$	$E_{5/2,1/2}$	$E_{5/2,-1/2}$	$E_{5/2,-3/2}$	$E_{5/2,-5/2}$	
∞	0	0	0	0	0	0	
	$E_{3,3}$	$E_{3,2}$	$E_{3,1}$	$E_{3,0}$	$E_{3,-1}$	$E_{3,-2}$	$E_{3,-3}$
∞	-1.0713	-0.001	0.03	0.93	0.03	-0.001	-1.0713
σ	0.0011	0.046	0.65	0.79	0.65	0.046	0.0011

	Energy	$E_{0,0}$	Δ_S				
∞	1.07149	1.0718	0.0000241				
σ	0.00006	0.0003	0.0000008				
	$E_{1/2,1/2}$	$E_{1/2,-1/2}$					
∞	0	0					
	$E_{1,1}$	$E_{1,0}$	$E_{1,-1}$				
∞	-0.012	1.665	-0.001				
σ	0.021	0.009	0.013				
	$E_{3/2,3/2}$	$E_{3/2,1/2}$	$E_{3/2,-1/2}$	$E_{3/2,-3/2}$			
∞	0	0	0	0			
	$E_{2,2}$	$E_{2,1}$	$E_{2,0}$	$E_{2,-1}$	$E_{2,-2}$		
∞	0.007	-0.05	1.61	-0.06	0.05		
σ	0.051	0.09	0.25	0.16	0.08		
	$E_{5/2,5/2}$	$E_{5/2,3/2}$	$E_{5/2,1/2}$	$E_{5/2,-1/2}$	$E_{5/2,-3/2}$	$E_{5/2,-5/2}$	
∞	0	0	0	0	0	0	
	$E_{3,3}$	$E_{3,2}$	$E_{3,1}$	$E_{3,0}$	$E_{3,-1}$	$E_{3,-2}$	$E_{3,-3}$
∞	-9.2	0.04	0.5	1.0	-0.3	-0.12	-0.26
σ	0.3	0.20	1.2	1.2	1.0	0.28	0.25

- Both have energy around 1.07149
- Most invariants (except $E_{3,\pm 3}$) are the same within errors
- Many invariants are exactly or asymptotically zero

- The first solution is real, the second only pseudo-real
- The second solution has asymmetric invariants
- It is similar to SL(2,C) solutions
- Its boundary state can be probably reached by (complex) marginal deformation of the first one

Summary and discussion

- We find real solutions reprenting Cardy boundary states
  - These solutions follow „selection rules“ regarding θ

$$\theta = \pm 2|J_i - J_f + l|\frac{\pi}{k}, \quad l \in \mathbb{Z}$$

  - Are there similar rules for BCFT results?

- Can we reach other θ?
  - A promising approach seems to be to fix the value of the marginal field
  - Combination of relevant and marginal deformations
  - Not yet clear how much of the moduli space is covered
  - Work in progress

- We also find pseudo-real solutions reprenting SL(2,C) boundary states
  - These solution follow „selection rules“ for θ
  - The other parameter ρ seems to be generic

- For $k \geq 3$ we find exotic solutions which describe boundary states breaking the current symmetry

- The number of these solutions is much smaller than in free boson on torus
  - Low number of relevant operators?
  - Do exotic boundary states typically have too high energy?
  - The condition $J_0^3|\Psi\rangle = 0$ could be too restrictive
  - The SU(2) symmetry could be fixed just using Z_2 subgroups of SU(2)
  - That would require a different ansatz for string field and new numerical algorithms

- Some of the exotic solutions we found could be related to analytic results
  - 0105038, 0705.1068