[based on arXiv:2003.05021 & related other works in progress]

Path-integral and quantum A_∞ structure of QFT

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Today's messages

Roughly speaking.. [Math aspects]

- Each quantum field theory that has the path-integral description

$$\text{correlation fnc.} \;\; \langle \dots \rangle = \int \mu_\phi \, (\dots) \qquad \text{e.g.} \quad \mu_\phi = \mathscr{D}\phi \, e^{S[\phi]}$$

always has own (quantum) A_∞ structure ν.

(For ordinary QFTs, this (quantum) A_∞ reduces to (quantum) L_∞ automatically.)

- Path-integral P always gives a morphism of such A_∞ str.

$$P \, \nu = \nu' \, P \qquad (P : \text{original } A_\infty \to \text{effective quantum } A_\infty)$$

Today's messages

Roughly speaking.. [Phys aspects]

- So, **as long as your original QFT is consistent,** all objects obtained by the path-integral also have own A_∞/L_∞ automatically!

  (e.g. effective theories, amplitudes, current recursion relations, or symmetries under RG flows)

- String field theory is a consistent UV finite theory:
  It gives typical examples to which you can apply these ideas "easily".

In this talk, I will explain these meanings more strictly.

Plan of talk

1. Why every path-integrable QFT have A_∞ / L_∞

2. Why the path-integral preserves such A_∞ / L_∞

(In particular, the perturbative path-integral gives A_∞/L_∞ morphisms very explicitly.)

I would like to emphasize that these 1 & 2 are mostly patch-work of ***known results.***

3. Application to perturbative SFT

1. Why every path-integrable QFT have A∞ / L∞

I told you that..

- Each quantum field theory that has the path-integral description

$$\text{correlation fnc.} \ \langle \dots \rangle = \int \mu_\phi (\dots) \qquad \text{e.g.} \quad \mu_\phi = \mathscr{D}\phi \, e^{S[\phi]}$$

always has own (quantum) A∞ structure ν.

- Let us explain the meanings of ***"consistent QFT, path-integrable QFT, or QFT that has the path-integral description"*** in this talk.

  That is ***"QFT solving the Batalin-Vilkovisky master equation"***.

1. Why every path-integrable QFT have A_∞ / L_∞

What was BV ?

- BV is a powerful and general formalism that enables us **to perform the path-integral, *even for gauge theory***. It is the geometry of the BV odd Laplacian $\boldsymbol{\Delta}$ with $(\boldsymbol{\Delta})^2=0$.

  To define $\int \mathscr{D}\phi \, (\dots)$, $\boldsymbol{\Delta}$-exacts vanish $\int \mathscr{D}\phi \, (\Delta \, \mathrm{exact} \,) = 0$ and the integrand must be

  $\boldsymbol{\Delta}$-closed : $\quad \Delta \, (\, \mathrm{integrand} \,) = 0$.

- Then, for each QFT, this consistency condition gives ***"the BV master equation".***

$$\Delta \, e^{S[\phi]} = 0 \iff \hbar \, \Delta \, S + \frac{1}{2} (\, S \, , S \,) = 0$$

(The BV bracket is defined by $(-)^A (A, B) \equiv \Delta(AB) - (\Delta A) B - (-)^A A (\Delta B)$).

1. Why every path-integrable QFT have A∞ / L∞

So, we consider QFT solving BV eq.

- The solution $S[\phi]$ of the BV master equation has the following form:

$$S[\phi] = \underbrace{S_{\mathrm{cl}}[\phi_{\mathrm{cl}}]}_{\text{classical action}} + \underbrace{\phi^* \,(S[\phi], c\,)}_{\text{for gauge degrees}} + \underbrace{c^* \,(S[\phi], \text{ghosts for ghosts})}_{\text{for redundancy of gauge degrees}} + \cdots$$

- This BV action $S[\phi]$ gives a set of "vertices" $\mu = \{\mu_n\}_{n>1}$ as follows

$$S[\phi] = \frac{1}{2}\langle \phi, \mu_1 \phi \rangle + \frac{1}{3}\langle \phi, \mu_2(\phi, \phi) \rangle + \frac{1}{4}\langle \phi, \mu_3(\phi, \phi) \rangle + \cdots$$

For a given QFT, this BV master action $S[\phi]$ is unique in some sense.
Actually, these multi-linear maps $\{\mu_1, \mu_n\}_{n>1}$ satisfy the (quantum) A∞/L∞ relations.

1. Why every path-integrable QFT have A∞ / L∞

Equivalent rep. of solving BV eq.

- We consider the operator $\hbar\,\Delta_S \equiv \hbar\,\Delta + (\,S\,,\quad)$ with $\Delta \equiv (-)^{\phi} \dfrac{\partial^2}{\partial\phi\partial\phi^*}$, which gives

$$\hbar\,\Delta_S\,\phi = -\frac{\partial S[\phi]}{\partial\phi} = \mu_1\phi + \mu_2(\phi,\phi) + \mu_3(\phi,\phi,\phi) + \cdots$$

- Note that *"solving BV eq."* equals to *"requiring $(\,\hbar\,\Delta_S\,)^2 = 0$"* because of

$$\big(\,\hbar\,\Delta_S\,\big)^2 = \Big(\,S\,,\hbar\,\Delta\,S + \frac{1}{2}(\,S\,,S\,)\Big)\ .$$

- Actually, as we see, $(\,\hbar\,\Delta_S\,)^2 = 0$ is nothing but the quantum A∞/L∞ .

1. Why every path-integrable QFT have A∞ / L∞

Solving BV eq. = requiring quantum A∞/L∞

- We can expand $(\hbar\,\Delta_S)^2 = 0$ acting on $\phi = \sum \phi_g + \sum \phi_g^*$ as follows

$$\big(\hbar\,\Delta_S\big)^2 \phi = \hbar\,\Delta_S \Big[\mu_1 \phi + \mu_2(\phi,\phi) + \mu_3(\phi,\phi,\phi) + \cdots \Big]$$

$$= \sum_n \Big[\hbar \sum_g (-)^g \frac{\partial^2}{\partial \phi_g \partial \phi_g^*}\, \mu_{n+2}(\phi,\ldots,\phi) \;+ \sum_{l+k=n} (-)^{\text{sign}}\, \mu_{l+1}\big(\ldots,\phi,\mu_k(\phi,\ldots,\phi)\,,\phi,\ldots\big) \Big]$$

- These are nothing but the A∞/L∞ relations, which may become more explicit if we use the symbols mimicking "complete basis of the inner product", $e_{-g} \equiv \dfrac{\partial \phi}{\partial \phi_g}$, $e_{1+g} \equiv \dfrac{\partial \phi}{\partial \phi_g^*}$, and expand each μ_n with respect to $\hbar$, such as $\mu_n = \mu_{n,[0]} + \hbar\,\mu_{n,[1]} + \hbar^2\,\mu_{n,[2]} + \hbar^3\,\mu_{n,[3]} + \cdots$.

1. Why every path-integrable QFT have A_∞ / L_∞

In summary..

- To have the path-integral, QFT must solve the BV master equation.

- "Solving BV eq. $\Delta\, e^{S[\phi]} = 0$" is the same as imposing the quantum A_∞ on vertices $\mu = \{\mu_1, \mu_n\}_{n>1}$ of your BV master action,

$$S[\phi] = \frac{1}{2}\langle \phi, \mu_1 \phi \rangle + \frac{1}{3}\langle \phi, \mu_2(\phi,\phi)\rangle + \frac{1}{4}\langle \phi, \mu_3(\phi,\phi)\rangle + \cdots$$

- ***So, each QFT has own intrinsic quantum A_∞/L_∞ arising from BV eq. .***

- This A_∞/L_∞ structure is **unique**, as is the proper BV master action.

2. Why the path-integral preserves A_∞ / L_∞

Next topic..

- We have noticed that ***every QFT have quantum A_∞ / L_∞ .***

- But, why does the path-integral preserve it ??

— That is also because of BV.

- It might be ***trivial***, as long as you can split $\phi = \phi' + \phi''$ and $\Delta = \Delta' + \Delta''$.

2. Why the path-integral preserves A∞ / L∞

As is well known..

- Any effective action $A[\phi']$ for "a given QFT $S[\phi' + \phi'']$ solving BV eq."

$$P : \; S[\phi' + \phi''] \; \longmapsto \; A[\phi'] \equiv \ln \int \mathscr{D}\phi'' \, e^{S[\phi' + \phi'']}$$

also solves the BV master equation : you quickly find $\int \mathscr{D}\phi'' \Delta''(\ldots) = 0$ and

$$\Delta' \, e^{A[\phi']} = \int \mathscr{D}\phi'' \, (\Delta' + \Delta'') e^{S[\phi' + \phi'']} = 0 \; .$$

- Hence, your BV effective QFT also has **own (quantum) A∞/L∞** , $\mu' = \{\mu'_n\}_n$.

The path-integral P preserves it in this sense : $P\,\mu = \mu'\,P$.

2. Why the path-integral preserves A_∞ / L_∞

Actually, these properties have been well used by experts.

- Flows of exact renormalization group with BV. [K.Costello 2007, R.Zucchini 2018]
- Realization of symmetry in ERG with BV. [Y.Igarashi, K.Itoh, H.Sonoda 2009]
- Combing BV and ERG. [T.Morris 2018, Y.Igarashi, K.Itoh, T.Morris 2019, P.Lavrov 2019]

— And there are other many earlier works..

Also, there are some works based on the A_∞/L_∞ side of BV

- BJ recursion relations of gluon, scattering amplitudes by using A_∞/L_∞.
  [M.Doubek et al 2017, B.Jurco, et al 2018, LT.Macrelli, et al 2019, A.S.Arvanitakis 2019, B.Jurco et al 2019]

Last year, the speaker studied **the classical (tree graphs) part** of the above result. He proposed how to reduce a given "covariant SFT" to corresponding "light-cone SFT". [HM. JHEP04(2019)143]

2. Why the path-integral preserves A∞ / L∞

Now, you may notice that effective quantum A_∞/L_∞ is trivial.

- We have learned that the path-integral preserves BV, and thus A_∞/L_∞ .
- So, as long as your original QFT is path-integrable,
  quantum A_∞/L_∞ structure of your effective QFT is automatic.
- But, is there any “explicit” construction of such a morphism ?

— We have it.

2. Why the path-integral preserves A_∞ / L_∞

"Explicit" construction of such P

- **The perturbative path-integral** gives such a morphism very **explicitly**.

  In other words, the Feynman graph expansion preserves quantum A_∞/L_∞ !!

- We noticed that the (non-perturbative) path-integral gives a morphism of BV,

$$P: \; S[\phi' + \phi''] \;\longmapsto\; A[\phi'] = \ln \int \mathscr{D}\phi''\, e^{S[\phi'+\phi'']} \quad ,$$

  which is often called a ERG transformation, and thus A_∞/L_∞ is automatic.

  The Feynman graph expansion of this **P** also gives a morphism.

2. Why the path-integral preserves A∞ / L∞

Can we obtain P directly in terms of A_∞ / L_∞?

- We obtained some results in terms of the BV master action $S[\phi] = S_{\mathrm{free}}[\phi] + S_{\mathrm{int}}[\phi]$.

  We can also obtain corresponding results in terms of A∞/L∞ more directly.

- It is given by "the homological perturbation $\mu_1 \mapsto \mu_1 + \mu_{\mathrm{int}} + \hbar\,\Delta$". By using coalgebra description, we get the effective quantum A∞/L∞ and morphism $P\,\mu = \mu'\,P$ directly.

  effective A∞/L∞ $\mu' = \mu_1' + i\,\mu_{\mathrm{int}}\,P$ & morphism $P = \dfrac{1}{1 + \mu_1^{-1}(\mu_{\mathrm{int}} + \hbar\,\Delta)}\,p$

  It is the same as the Feynman graph expansion, or applying Wick's theorem, for

  the ERG transformation $P:\ S[\phi' + \phi''] \longmapsto A[\phi'] = \ln \displaystyle\int \mathscr{D}\phi''\, e^{S[\phi'+\phi'']}$.

  (Sorry, we skip "what the homological perturbation was" now, which is in appendix.)

2. Why the path-integral preserves A_∞ / L_∞

Comments on path-integral by homological perturbation

- The perturbative path-integral, or the Feynman graph expansion, can be obtained as a result of the homological perturbation of $\hbar\,\Delta_{S_{\text{int}}}$, and thus it preserves BV eq. and A_∞/L_∞ .

Following perturbations give the Wick's theorem and the perturbative path-integral :

$$(S_{\text{free}}\,,\quad) \longmapsto \underbrace{\overbrace{\hbar\,\Delta}^{\text{perturbation}} + (S_{\text{free}},\quad)}_{\text{Wick's theorem}} \longmapsto \hbar\,\Delta_S \equiv \underbrace{\overbrace{\hbar\,\Delta + (S_{\text{int}},\quad)}^{\text{full perturbation}} + (S_{\text{free}},\quad)}_{\text{Perturbative path integral}}$$

- It is the same as the Feynman graph but can give explicit constructions of some quantities.

(We may review these facts later, if we have enough time before *our cut-off*.)

2. Why the path-integral preserves A_∞ / L_∞

Some comments

- We considered algebraic aspects only, but now it will be trivial to you.
  Please note that all physically important informations are in your *concrete construction of "regular" propagators*. (You might learn it from D-instanton.)

- In general, it may be a challenging problem to solve the BV master equation for QFT with finite cut-off, if QFT is not UV finite.

  Solving BV for QFT without cut-off is not difficult, but "regular propagators" will require cut-off dependence.
  As I know, even for Yang-Mills, we know a 1-loop level BV master action only. [Y.Igarashi, K.Itoh, T.Morris 2019]

- In general, your BV Laplacian will have cut-off dependence and then **ERG flows** are given by BV canonical transformations, or morphisms of quantum A_∞/L_∞.
  ERG flows shift the cut-off dependence of your BV Laplacian.

- So, application to SFT is easier than other UV divergent QFTs and it is exact.

3. Application to perturbative SFT

We now consider SFT

- As is known, ***SFT is a consistent UV finite QFT***, which satisfies $\Delta\, e^{S[\phi]} = 0$.

  For a given master action $S[\phi] = \frac{1}{2}\langle \phi, \mu_1 \phi \rangle + \frac{1}{3}\langle \phi, \mu_2(\phi, \phi)\rangle + \frac{1}{4}\langle \phi, \mu_3(\phi, \phi, \phi)\rangle + \cdots$,

  we can consider $\phi = \phi' + \phi''$ and *the perturbative path-integral of* ϕ'' as follows

$$P \;:\; S[\phi' + \phi''] \;\longmapsto\; A[\phi'] = \ln \int \mathscr{D}\phi''\, e^{S[\phi' + \phi'']}$$

- Then, ***thanks to BV***, *the quantum A_∞/L_∞ of your effective action* is *automatic*

$$A[\phi'] = \frac{1}{2}\langle \phi', \mu_1' \phi' \rangle + \frac{1}{3}\langle \phi', \mu_2'(\phi', \phi')\rangle + \frac{1}{4}\langle \phi, \mu_3(\phi', \phi', \phi')\rangle + \cdots$$

3. Application to perturbative SFT

App.1) Typical Examples of $\phi = \phi' + \phi''$

- Your effective theory $A[\phi']$ has A∞/L∞ corresponding to ***the splitting*** $\phi = \phi' + \phi''$ because it changes ***the propagators*** $(\mu_1'')^{-1}$ *given by* $\mu_1 = \mu_1' + \mu_1''$,

$$A[\phi'] \equiv \ln \int \mathscr{D}\phi''\, e^{S[\phi'+\phi'']} = \frac{1}{2}\langle \phi', \mu_1' \phi' \rangle + \frac{1}{3}\langle \phi', \mu_2'(\phi', \phi')\rangle + \frac{1}{4}\langle \phi, \mu_3(\phi', \phi', \phi')\rangle + \cdots$$

- Typical examples : (1) As usual, $\phi = \phi'_{\mathrm{IR}} + \phi''_{\mathrm{UV}}$ gives Wilsonian with A∞/L∞ .

  (2) $\phi = \phi'_{\text{on shell}} + \phi''_{\text{off shell}}$ gives the S-matrix $A[\phi']$ as a minimal model of A∞/L∞ .

  (3) $\phi = \phi'_{\text{massless}} + \phi''_{\text{massive}}$ gives A∞/L∞ effective QFT $A[\phi']$ with finite α' .

  (4) $\phi = \phi'_{\text{phys}} + \phi''_{\text{gauge+unphys}}$ gives "gauge-removed" SFT with A∞/L∞ .

3. Application to perturbative SFT

App.2) Light-cone reduction : a special choice of $\phi = \phi'_{\text{phys}} + \phi''_{\text{gauge+unphys}}$

- For a given covariant SFT, there exists the corresponding light-cone SFT.
- The BRST operator of (super) strings has the similarity transformation, for example,

$$Q = e^{-R} \Big(\overbrace{c_0 L_0}^{\mu'_1} - p^+ \overbrace{\sum_{n \neq 0} c_{-c}\, a_n^+}^{\mu''_1} \Big) e^R \qquad \text{(open strings) .}$$

[Aisaka, Kazama 2004] for bosonic
[Kazama, Yokoi 2011] for super

It induces $\phi_{\text{covariant}} = \phi'_{\text{light cone}} + \phi''_{a^\pm, b, c}$ and the A_∞ / L_∞ light-cone SFT :

$$A[\phi'] = \underbrace{\frac{1}{2}\langle \phi', c_0 L_0^{\text{lc}}\, \phi' \rangle + \sum_n \frac{1}{n+1} \langle \phi', \mu_n(\phi', \dots, \phi') \rangle}_{\textit{same form as the covariant SFT}} + \underbrace{\sum_{g,n} \frac{\hbar^g}{n+1} \langle \phi, \mu'_{n,[g]}(\phi', \dots, \phi') \rangle}_{\textit{effective vertices}}$$

"When effective vertices vanish and it reduces to Kaku-Kikkawa's theory" will be reported by [corroboration with Ted Erler] .

3. Application to perturbative SFT

App.3) Realization of symmetry as A∞/L∞

- Recall that *composite operators of symmetry* $\delta_{\rm sym}\phi = \mathcal{O}_{\rm sym}[\phi]$ survive along **ERG flows**

$$\int \mathcal{D}\phi \, \mathcal{O}_{\rm sym}[\phi] \, e^{S[\phi]} = \int \mathcal{D}\phi' \, \mathcal{O}'_{\rm sym}[\phi'] \, e^{A[\phi']}$$

[Review: Y.Igarashi et al 2009]

and there is *no loss of symmetry*, although their forms drastically change along flows.

$$\delta_{\rm sym}\phi = \mathcal{O}_{\rm sym}[\phi] \quad \longmapsto \quad \delta_{\rm sym}\phi' = \mathcal{O}'_{\rm sym}[\phi']$$

- This is also true for our case. The relation between $\mathcal{O}_{\rm sym}[\phi]$ and $\mathcal{O}'_{\rm sym}[\phi']$ is explicit. It is given by a morphism of A∞/L∞ :

$$\mathcal{O}'_{\rm sym}[\phi'] = P\left(\, \mathcal{O}_{\rm sym}[\phi] \, \right)$$

[work in progress]

So, symmetry of the original QFT also exists in your effective QFT in terms of A∞/L∞ , though it may take some highly-nonlinear form. (e.g. Lorentz generators in light-cone SFT.)

Summary

As we saw, in QFT, quantum A_∞/L_∞ is always there..

- QFT has own quantum $A\infty/L\infty$, which is equivalent to solving BV.
- The path-integral preserves it, and thus your effective A_∞/L_∞ is **automatic**.
- Symmetry in effective QFT is also encoded into A_∞/L_∞ .

Comments

- We learned that QFT and A_∞/L_∞ are in one-to-one, thanks to BV.
  It may imply that

  "deformation of QFT" and "deformation of A_∞/L_∞" are in one-to-one.

  — That is given by quantum open-closed homotopy algebra or $IBL\infty$.

Thank you for your attention !!

5-slides Review :
Path-integral by Homological perturbation

2. Why the path-integral preserves A∞ / L∞

Review : path-integral = homological perturbation

- The perturbative path-integral, or the Feynman graph expansion, can be obtained as a result of the homological perturbation of $\hbar\,\Delta_{S_{\mathrm{int}}}$, and thus it preserves BV eq. and A∞/L∞ .

  — Let us review this fact in more detail.

- The free BV theory $S_{\mathrm{free}}[\phi] = \frac{1}{2}\langle\, \phi\,, \mu_1\, \phi\,\rangle$ solves $(\, S_{\mathrm{free}}\,, S_{\mathrm{free}}\,) = 0\,, \quad \Delta\, S_{\mathrm{free}} = 0\,.$

  Its equations of motion is, classically, given by $\mu_1\, \phi = (\, S_{\mathrm{free}}\,, \phi\,) = 0\,.$

- This $(\, S_{\mathrm{free}}\,, \quad)$ is a nilpotent operator, whose cohomology is the classical on-shell.

2. Why the path-integral preserves A∞ / L∞

Review : path-integral = homological perturbation

- The relation of “classical” off-shell and on-shell is described by the deformation retract, homotopy equivalent pairs of **(vector space , differential)** , as follows

$$\Big(\, \text{state space}\,,\ \underbrace{(\, S_{\text{free}}\,,\quad)}_{\text{differential } \hat{\mu}_1}\,\Big) \ \underset{i}{\overset{p}{\longleftrightarrow}}\ \underbrace{\Big(\, \text{on shell}\,,\ 0\,\Big)}_{\text{cohomology of } \hat{\mu}_1}$$

- The BV propagator $\hat{\mu}_1^{-1}$ gives a Hodge decomposition : $\hat{\mu}_1\,\hat{\mu}_1^{-1} + \hat{\mu}_1^{-1}\,\hat{\mu}_1 = 1 - ip$.

- Now, because of $\Delta\, S_{\text{free}} = 0$, we can consider the homological perturbation of

$$(S_{\text{free}}\,,\quad) \longmapsto \hbar\,\Delta_{S_{\text{free}}} \equiv \underbrace{\hbar\,\Delta}_{\text{perturbation}} + (S_{\text{free}},\quad)$$

2. Why the path-integral preserves A_∞ / L_∞

Review : path-integral = homological perturbation

- As a result, we get a new deformation retract

$$\Big(\,\text{state space}\,,\ \underbrace{\hbar\,\Delta + (\,S_{\text{free}}\,,\quad)}_{\text{new differential}}\,\Big) \ \underset{I}{\overset{P}{\longleftrightarrow}}\ \underbrace{\big(\,\text{on shell}\,,\ 0\,\big)}_{\text{cohomology of }\hbar\,\Delta_{S_{\text{free}}}}$$

- The homological perturbation lemma tells us that the morphism P is given by solving the recursive relation $P = p + \hbar\,\hat{\mu}_1^{-1}\Delta\,P$, which gives P explicitly.

Actually, this $P = \dfrac{1}{1+\hbar\,\hat{\mu}_1^{-1}\Delta}\,p$ is nothing but the Feynman graph expansion.

2. Why the path-integral preserves A∞ / L∞

Review : path-integral = homological perturbation

- The commutator of $\hat{\mu}_1^{-1} \equiv (S_{\rm free}, \quad)^{-1}$ and $\hbar\,\Delta$ is proportional to $\mu_1^{-1} \dfrac{\partial^2}{\partial\phi_g \partial\phi_{-g}}$

- You can expand $P = \dfrac{1}{1 + \hbar\,\hat{\mu}_1^{-1}\Delta} p$ acting on $e^{S_{\rm int}[\phi]}$ as follows

$$P\,e^{S_{\rm int}[\phi]} = \sum_n (\hbar\,\hat{\mu}_1^{-1}\Delta)^n p\,e^{S_{\rm int}[\phi]} = \exp\Big[\frac{1}{2}\sum_g \mu_1^{-1}\frac{\partial^2}{\partial\phi_{-g}\partial\phi_g}\Big]\,e^{S_{\rm int}[\phi]}$$

- It is the same as the Wick's theorem given by Gaussian $\displaystyle\int \mathscr{D}\phi\, e^{\frac{1}{2}\phi\,\mu_1\,\phi} = 1$.

2. Why the path-integral preserves A∞ / L∞

Review : path-integral = homological perturbation

- So, the Feynman graph expansion is given by the homological perturbation :

$$(S_{\text{free}}\,,\quad) \longmapsto \hbar\,\Delta_{S_{\text{free}}} \equiv \underbrace{\hbar\,\Delta}_{\text{perturbation}} + (S_{\text{free}},\quad)$$

- By the way, what kind of P_{int} does **the full perturbation** gives ?

$$(S_{\text{free}}\,,\quad) \longmapsto \hbar\,\Delta_{S_{\text{free}}} \longmapsto \hbar\,\Delta_{S} \equiv \underbrace{\hbar\,\Delta + (S_{\text{int}},\quad)}_{\text{full perturbation}} + (S_{\text{free}},\quad)$$

— That is the “normalized” perturbative path-integral $P_{\text{int}}(\ldots) = Z^{-1}\int \mathscr{D}\phi\,(\ldots)e^{S_{\text{free}}[\phi]+S_{\text{int}}[\phi]}$.