Weak Coupling Expansion of Lieb-Liniger Equation

-Collective Valiable Description of Bose Systems-

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1 Many boson system with two-body force

1.1 Original Hamiltonian and collective variables

$$\mathcal{H} = \sum_k k^2 a_k^\dagger a_k + \frac{1}{2\Omega} \sum_{k,l,l'} v(k) a_l^\dagger a_{l'}^\dagger a_{l'+k} a_{l-k}.$$

Here $a_k^\dagger$ and a_k are creation and annihilation operators of bosons with momentum k, respectively. Ω is the volume of the system. For simplicity we put $\hbar = 1$ and $2m = 1$. At the ground state it is expected that almost all bosons are at zero momentum state. In Bogoliubov's theory in 1947 $a_0^\dagger \simeq a_0 \simeq \sqrt{N_0}$. He got the excitation spectrum with momentum k as $\epsilon_k = \sqrt{k^2(k^2 + \frac{2N_0}{\Omega} v(k))}$. Much effort has been made to treat this Hamiltonian by means of collective variables such as density, phase and velocity.

1. Sunakawa, Yamasaki and Kebukawa (SYK) theory by density and velocity operators.

2. Bogoliubov and Zubarev (BZ) theory (1955) by density variable.

3. Correlated basis function (CBF) method by Lee et al.

1.2 BZ Hamiltonian

$$H = H_0 + \frac{1}{\sqrt{N}} H_1,$$

$$H_0 = E^0 + \sum_{k \neq 0} \frac{k^2}{\lambda_k} b_k^\dagger b_k,$$

$$H_1 = \frac{1}{4} \sum_{k,k'} (k, k') \sqrt{\frac{\lambda_{k+k'}}{\lambda_k \lambda_{k'}}} (b_{k+k'} + b^\dagger_{-k-k'})((1+\lambda_k) b_{-k} + (\lambda_k - 1) b_k^\dagger)((1+\lambda_{k'}) b_{-k'} + (\lambda_{k'} - 1) b_{k'}^\dagger),$$

$$E^0 = \frac{N(N-1)}{2\Omega} \nu(0) + \frac{1}{2} \sum_{k \neq 0} \left(k^2/\lambda_k - k^2 - \frac{N}{\Omega} \nu(k) \right), \ \lambda_k \equiv \sqrt{k^2/(k^2 + 2N\nu(k)/\Omega)}.$$

1.3 SYK Hamiltonian

$$H = H_0 + \frac{1}{\sqrt{N}} H_1 + \frac{1}{N} H_2 + \frac{1}{N^{3/2}} H_3 + ...,$$

$$H_1 = \frac{1}{4} \sum_{p,q,p+q\neq 0} (pq) \sqrt{\frac{\lambda_p \lambda_q}{\lambda_{p+q}}} \{(1 + \lambda_p \lambda_q)(b_p^\dagger b_{-p-q}^\dagger b_q^\dagger + b_p b_{-p-q} b_q$$

$$+ b_P^\dagger b_q^\dagger b_{p+q} + b_{p+q} b_p b_q) - 2(1 - \lambda_p \lambda_q)(b_{-q}^\dagger b_{p+q}^\dagger b_p + b_p^\dagger b_{p+q} b_{-q})\},$$

$$H_n = \frac{(-1)^{n-1}}{4} \sum_{p_1, p_2, ..., p_{n+2}, p_1 + ... p_{n+2} = 0} (p_1 p_{n+2})(\lambda_{p_1} \lambda_{p_2} ... \lambda_{p_{n+2}})^{1/2} (b_{p_1} + b_{-p_1}^\dagger)...(b_{p_{n+2}} + b_{-p_{n+2}}^\dagger),$$

for $n \geq 2$.

2 Perturbation expansion of ground state energy

Zero-th term of ground state energy is the same. But for the next term BZ and SYK is the sama but CBF give different formula

$$E^2_{BZ,SYK} = \frac{1}{24}\sum_{p,q,r\neq 0} \delta_{p+q+r,0}\lambda_p\lambda_q\lambda_r(p^2/\lambda_p + q^2/\lambda_q + r^2/\lambda_r$$

$$-\frac{[(pq)(1+\frac{1}{\lambda_p\lambda_q}) + (qr)(1+\frac{1}{\lambda_q\lambda_r}) + (rp)(1+\frac{1}{\lambda_r\lambda_p})]^2}{p^2/\lambda_p + q^2/\lambda_q + r^2/\lambda_r}),$$

$$E^2_{CBF} = \frac{1}{24}\sum_{p,q,r\neq 0} \delta_{p+q+r,0}[(1-\lambda_p)(1-\lambda_q)(1-\lambda_r)(p^2+q^2+r^2) - \lambda_p\lambda_q\lambda_r$$

$$\times\frac{[(pq)(1-\lambda_p^{-1})(1-\lambda_q^{-1}) + (qr)(1-\lambda_q^{-1})(1-\lambda_r^{-1}) + (rp)(1-\lambda_r^{-1})(1-\lambda_p^{-1})]^2}{p^2/\lambda_p + q^2/\lambda_q + r^2/\lambda_r}.$$

3 Lieb-Liniger equation for 1d δ-function bosons

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1\le i<j\le N} \delta(x_i - x_j).$$

The ground state energy per unit length in the thermodynamic limit is given by Lieb Liniger equation (1963) for the density $\rho(k)$ of quasi momenta.

$$\rho(k) - \frac{c}{\pi}\int_{-k_0}^{k_0} \frac{\rho(k')}{(k-k')^2 + c^2}\mathrm{d}k' = \frac{1}{2\pi}, \quad c > 0.$$

The density of bosons and density of energy per unit length are given by

$$n = \int_{-k_0}^{k_0} \rho(k)\mathrm{d}k, \quad e_0 = \int_{-k_0}^{k_0} k^2\rho(k)\mathrm{d}k.$$

This case is $\Omega = L$ and $d = 1$ and $\nu(k) = 2c$. One can calculate $E^2_{SYK,BZ}$ and E^2_{CBF} analytically. Details are in my paper in 1975.[2]

Using Bogoliubov perturbation theory Lieb-Liniger predicted

$$e_0 = n^3 u(c/n), \quad u(\gamma) = \gamma - \frac{4}{3\pi}\gamma^{3/2} + o(\gamma^{3/2}), \quad \gamma \to 0+.$$

The third term is more controversial. I calculated the third term of $u(\gamma)$ using Bogolieubov -Zubarev theory as

$$(\frac{1}{12} - \frac{1}{\pi^2})\gamma^2 = -0.0179878503090044\gamma^2.$$

But the correlated basis function method(CBF) gave

$$(\frac{1}{6} - \frac{1}{\pi^2})\gamma^2 = 0.065345483024328\gamma^2.$$

My numerical calculation of LL equation gave

$$u(\gamma) \simeq \gamma - \frac{4}{3\pi}\gamma^{3/2} + 0.0654\gamma^2 - 0.0018\gamma^{5/2}.$$

Then I concluded that CBF method gave the correct third term(1975). Recently Kaminaka and Wadati (2011) concluded the third term should be $(\frac{1}{8} - \frac{1}{\pi^2})\gamma^2 = 0.0236788163576622\gamma^2$. Then the third term is still controversial.

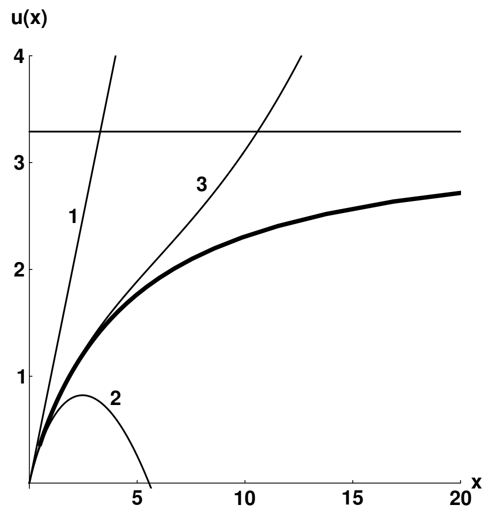

Figure 1: The function $u(x)$ determines the zero temperature properties of delta-function Bose gas. Thick line is the result of numerical calculation of Lieb-Liniger equation. Line 1 is the result of primitive perturbation. Line 2 is the result of Bogoliubov theory up to $\gamma^{3/2}$. Line 3 is the perturbation result up to γ^2 by the author[2].

4 Condenser Problem of Circular Disks

Consider the problem of two conducting circular disks with diameter 1 and the distance of two disks is κ. They are inversely charged. The charge distribution functions are $\sigma(r)$ and $-\sigma(r)$. Voltage difference is assumed to be V_0. The equation for $\sigma(r)$ is

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^1\Big(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}-\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta+\kappa^2}}\Big)\sigma(r')r'\mathrm{d}r'\mathrm{d}\theta = V_0,$$

or

$$\frac{2}{\pi}\int_0^1\Big[\frac{1}{r+r'}K\Big(\frac{2rr'}{(r+r')^2}\Big)-\frac{1}{\sqrt{(r+r')^2+\kappa^2}}K\Big(\frac{2rr'}{(r+r')^2+\kappa^2}\Big)\Big]\sigma(r')r'\mathrm{d}r' = V_0,$$

$$\text{at } \ 1 \geq r \geq 0.$$

This equation looks quite different from LL equation. But if we use Abel transformation

$$f(x) = 2\pi \int_x^1 \frac{r\sigma(r)}{\sqrt{r^2 - x^2}} \mathrm{d}r$$

we have Love's equation(1949)

$$f(x) - \frac{\kappa}{\pi} \int_{-1}^1 \frac{f(y)}{(x-y)^2 + \kappa^2} \mathrm{d}y = V_0.$$

This is much earlier than Lieb-Liniger(1963).
Inverse of Abel transform is

$$\sigma(r) = -\int_r^\infty \frac{df(y)}{dy} \frac{1}{\sqrt{y^2 - r^2}} dy.$$

If we put $V_0 = 1/2\pi$ and $\kappa = c/k_0$, two equations are completely the same.

Tracy and Widom's analysis(2016)

$$C = \frac{\kappa}{\gamma} = \pi \int_0^1 r\sigma(r)dr, \quad u(\gamma) = \frac{\pi}{2C^3} \int_0^1 r^3\sigma(r)dr.$$

$$C = \frac{1}{4\kappa}\Big[1 - \frac{\kappa}{\pi}(1 + \log\frac{\kappa}{16\pi}) + \frac{\kappa^2}{4\pi^2}(\log^2(\frac{\kappa}{16\pi}) - 2) + o(\kappa^2)\Big].$$

$$\pi \int_0^1 r^3\sigma(r)dr = C - \frac{D_1}{2}.$$

$$D_1 = \frac{1}{4\kappa} + \frac{2}{3\pi} + \frac{\kappa}{\pi^2}[-\frac{\pi^2}{6} - \frac{1}{4} + \frac{1}{4}\log^2\frac{\kappa}{16\pi} - \frac{3}{2}\log\frac{\kappa}{16\pi}] + o(\kappa).$$

Then we have

$$u = \frac{C - D_1/2}{2C^3} = 4\kappa^2 \frac{1 + \frac{\kappa}{\pi}(-2\log\frac{\kappa}{16\pi} - \frac{14}{3}) + \frac{\kappa^2}{\pi^2}(\frac{3}{2}\log^2\frac{\kappa}{16\pi} + 6\log\frac{\kappa}{16\pi} + \frac{2\pi^2}{3}) + o(\kappa^2)}{[1 - \frac{\kappa}{\pi}(1 + \log\frac{\kappa}{16\pi}) + \frac{\kappa^2}{4\pi^2}(\log^2(\frac{\kappa}{16\pi}) - 2) + o(\kappa^2)]^3}.$$

On the other hand γ and κ have the following relation,

$$\gamma = \frac{4\kappa^2}{1 - \frac{\kappa}{\pi}(1 + \log\frac{\kappa}{16\pi}) + \frac{\kappa^2}{4\pi^2}(\log^2(\frac{\kappa}{16\pi}) - 2) + o(\kappa^2)}.$$

Inverse of this relation is

$$\kappa = a_0\gamma^{1/2} + a_1\gamma\log\gamma + a_2\gamma + a_3\gamma^{3/2}(\log\gamma)^2 + a_4\gamma^{3/2}\log\gamma + a_5\gamma^{3/2} + ...,$$

with

$$a_0 = \frac{1}{2},\ a_1 = -\frac{1}{16\pi},\ a_2 = \frac{\log 32\pi - 1}{8\pi}, a_3 = \frac{1}{128\pi^2},\ a_4 = \frac{1 - \log 32\pi}{32\pi^2},$$

$$a_5 = \frac{1 - 4\log 32\pi + 2\log^2 32\pi}{64\pi^2}.$$

Using these, we have

$$u(\gamma) = \gamma - \frac{4}{3\pi}\gamma^{3/2} \underline{+\left(\frac{1}{6} - \frac{1}{\pi^2}\right)\gamma^2} + o(\gamma^2).$$

All logarithmic terms disappear! My result in 1975 is reproduced.

5 Next terms of weak coupling expansion

Recently Prolhac did a precise numerical analysis of Lieb-Liniger equation and obtained the following results,

$$u(\gamma) \simeq \gamma - \frac{4}{3\pi}\gamma^{3/2} + a\gamma^2 + b\gamma^{5/2} + c\gamma^3 + d\gamma^{7/2} + e\gamma^4.$$

$$a = \frac{1}{6} - \frac{1}{\pi^2} = (\zeta(2) - 1)/\pi^2 = 0.0653454830$$

$$b = (\zeta(3)\frac{3}{8} - \frac{1}{2})/\pi^3 = -0.0015876998655 \quad ????$$

$$c = (-\frac{45\zeta(5)}{1024} + \frac{15\zeta(3)}{256} - \frac{1}{32})/\pi^5 = -0.000020864 \quad ???$$

$$d = -\zeta(3)/\pi^4 = -0.01234029 \quad ????$$

6 Conclusion

- $$\lim_{n,L\to\infty} \frac{E^0}{L} = n^3(\gamma - \frac{4}{3\pi}\gamma^{3/2}),$$
  $$\lim_{n,L\to\infty} \frac{E^2_{BZ,SYK}}{L} = n^3\gamma^2(\frac{1}{12} - \frac{1}{\pi^2}) = -0.01798785 n^3\gamma^2,$$
  $$\lim_{n,L\to\infty} \frac{E^2_{CBF}}{L} = n^3\gamma^2(\frac{1}{6} - \frac{1}{\pi^2}) = 0.065345483024328 n^3\gamma^2.$$
  Then BZ theory and SYK theory failed to give the ground state energy at the second order. CBF method seems to give the correct second order energy.

- Results of condenser problem of circular disks also support my calculation

- At the present stage, systematic weak coupling expansion method for LL equation is not known!

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