Quantum quenches in the XXZ chain via six-vertex model with domain wall boundary conditions

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Workshop: Exactly Solvable Quantum Chains, Natal 2018

Outline

1. Inhomogeneous Quantum Quenches
2. An exact formula for the return probability
3. Discussion

Work based on

JMS [1707.06625, J. Stat. Mech 2017]

N. Allegra, J. Dubail, JMS, J. Viti [1512.02872, J. Stat. Mech 2016]

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JMS [1707.06625, J. Stat. Mech 2017]

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Jérôme Dubail (Nancy)

Jacopo Viti (Natal)

Quantum quenches

Prepare a system in some pure state $|\Psi_0\rangle$.

Evolve with the Hamiltonian H, $|\Psi_0\rangle$ not an eigenstate.

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

- Unitary evolution, no coupling to an environment.
- Cold atom systems. Peculiar thermalization properties.

Quench studied here

Initial state $|\Psi_0\rangle = |\ldots \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \ldots\rangle$

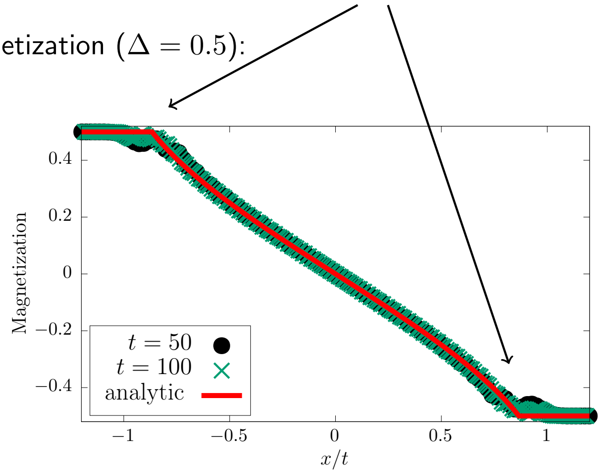
Time evolution $|\Psi(t)\rangle = e^{-itH_{XXZ}} |\Psi_0\rangle$

$$H_{XXZ} = \sum_{x \in \mathbb{Z}+1/2} \left(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta S_x^3 S_{x+1}^3 \right)$$

Free fermion case $(\Delta = 0)$ [Antal, Rácz, Rákos, and Schütz, 1999]

Interactions $(\Delta \neq 0)$: MPS numerical techniques
[Gobert, Kollath, Schollwöck, and Schütz 2005]

- Hydrodynamic regime: large x, large t, finite x/t.
- Finite speed of propagation: light cone.
- Magnetization ($\Delta = 0.5$):



Effective descriptions

- Generalized hydrodynamics ($|\Delta| < 1$, ballistic)
  [Castro-Alvaredo, Doyon, Yoshimura 2016]
  [Bertini, Collura, De Nardis, Fagotti 2016]

  This particular quench (root of unity $\Delta = \cos\frac{\pi p}{q}$),

$$S_x^3(x/t) = -\frac{q}{2\pi}\arcsin\left(\frac{\sin\frac{\pi}{q}}{\sin\frac{\pi p}{q}}\frac{x}{t}\right) \qquad (!)$$

  [De Luca, Collura, Viti 2017]

- $\Delta = 1$ is an open problem. Super diffusive behavior ($t^{3/5}$ here) was conjectured from numerics [Ljubotina, Znidaric, Prosen 2017]

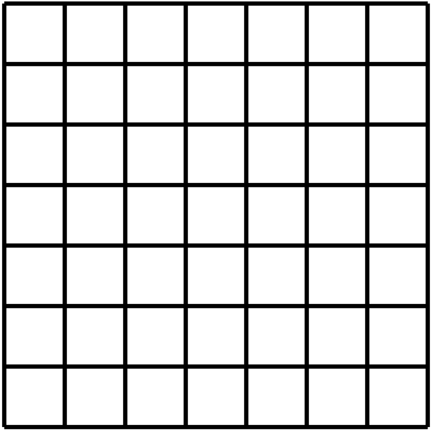
I compute the return probability $\mathcal{R}(t) = |\langle\Psi(0)|\Psi(t)\rangle|^2$ exactly $\forall t$

[JMS 2017]

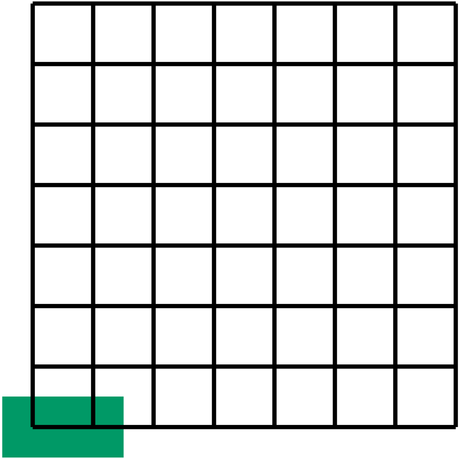
Simple guess for asymptotics: ballistic, so $\mathcal{R}(t) \lesssim e^{-at}$.

Nb: $\overline{\mathcal{R}(t)} \sim \prod_{k=1}^{\infty} \left(1 - e^{-2k\eta}\right)^2 \quad , \quad \cosh\eta = \Delta > 1$

[Mossel, Caux 2011]



Fun with dimers



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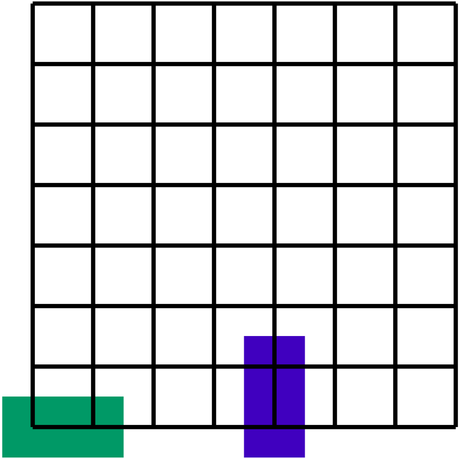

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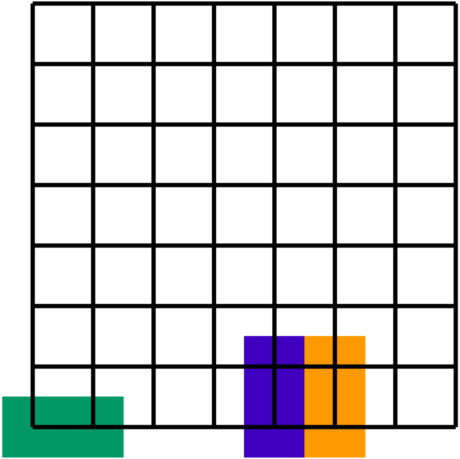

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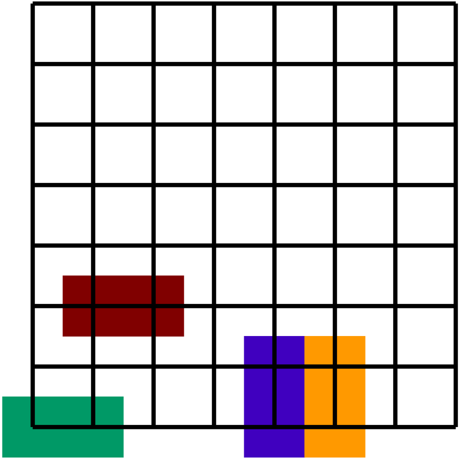

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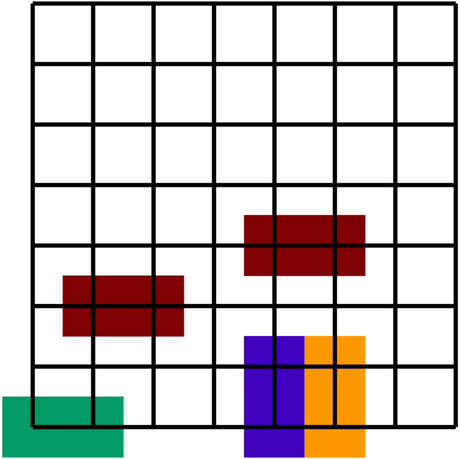

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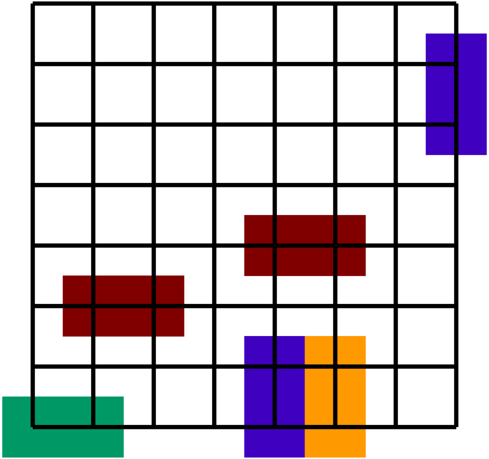

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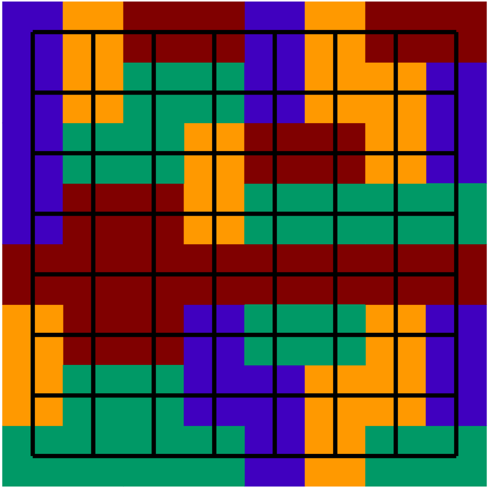

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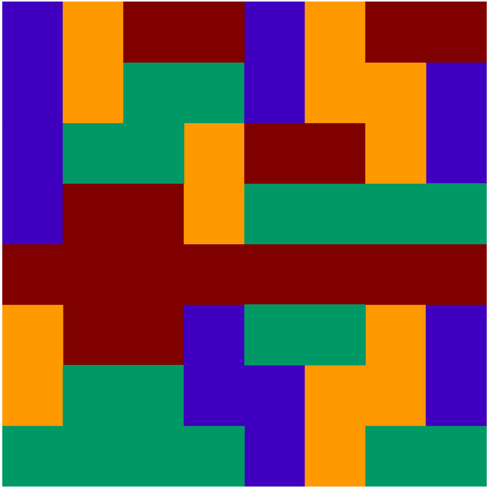

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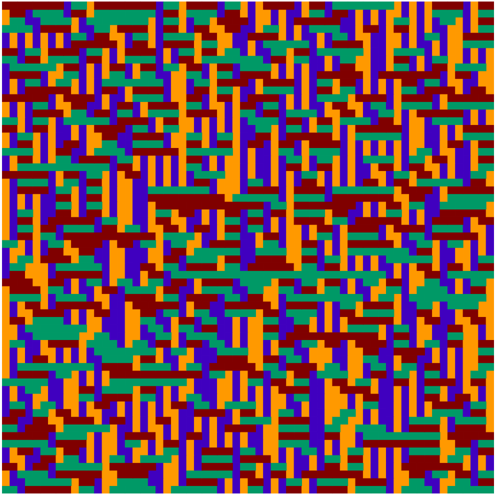

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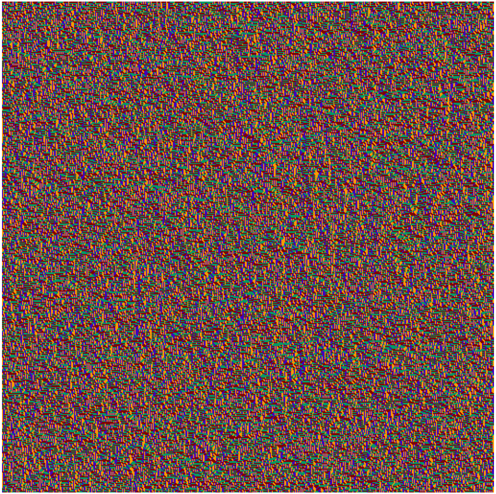
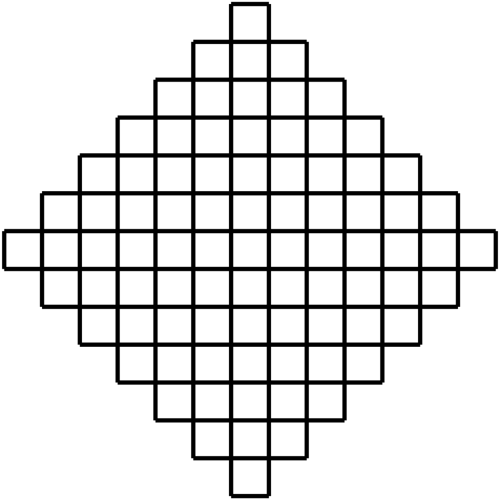


Long range correlations: gaussian free field, or coulomb gas, or free compact boson CFT ($c = 1$), or euclidean Luttinger liquid.

Dimer coverings on the Aztec diamond



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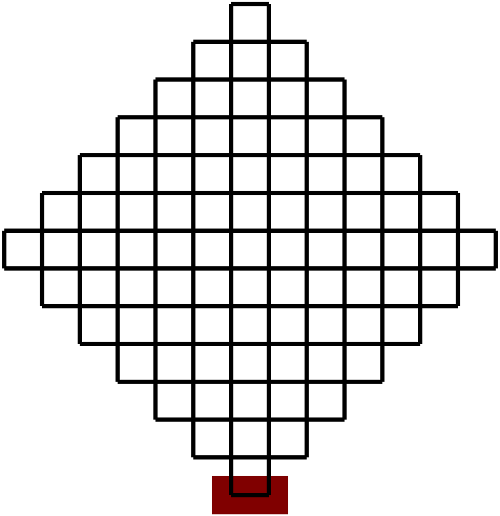

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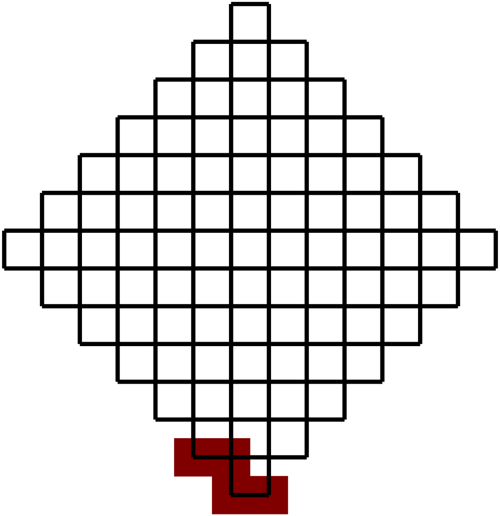

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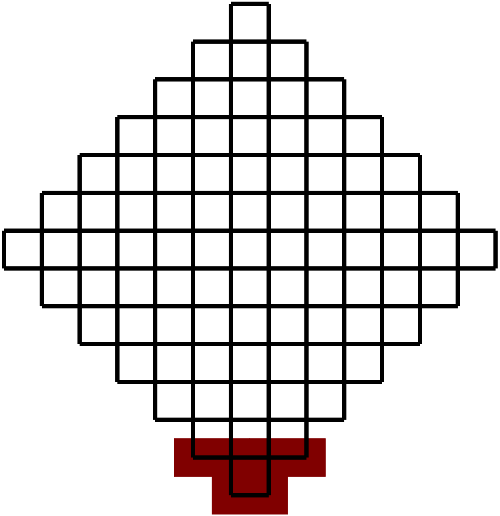

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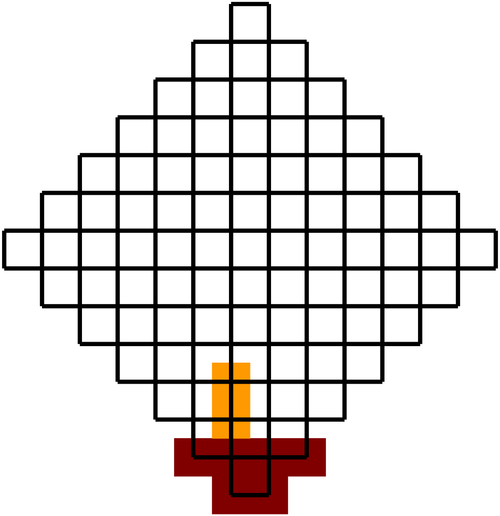

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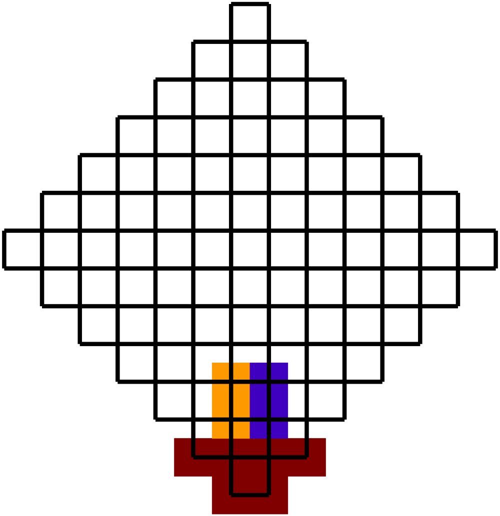

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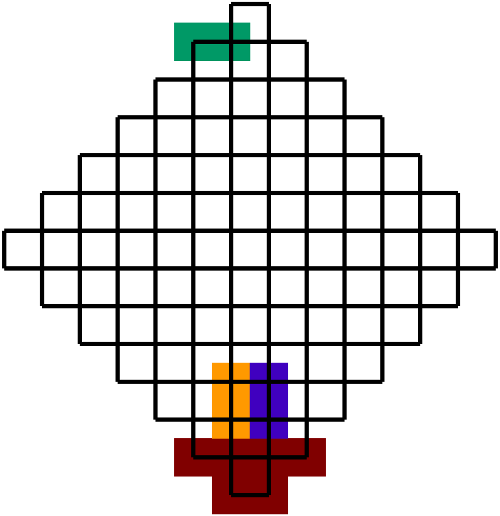

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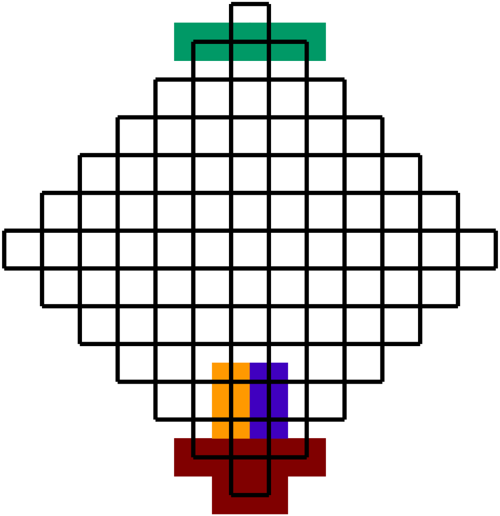

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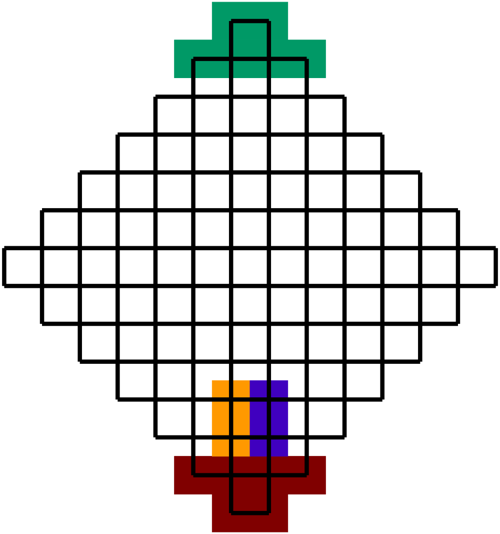

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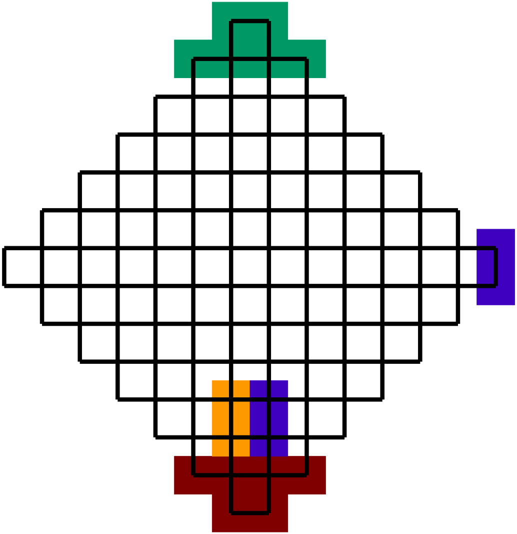

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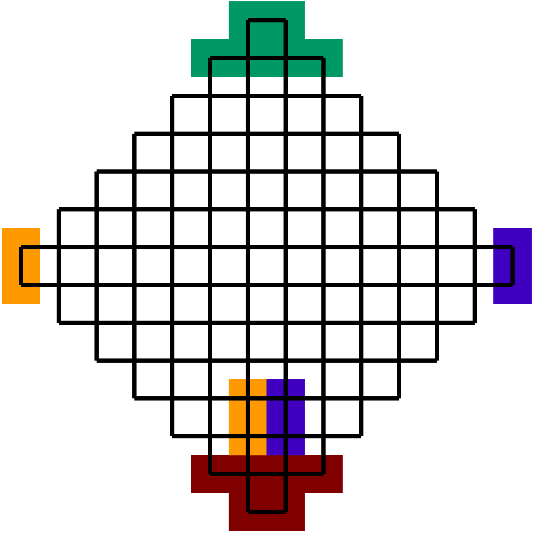

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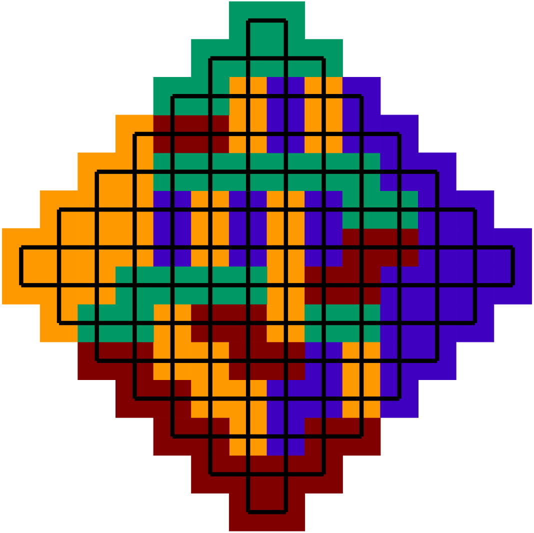

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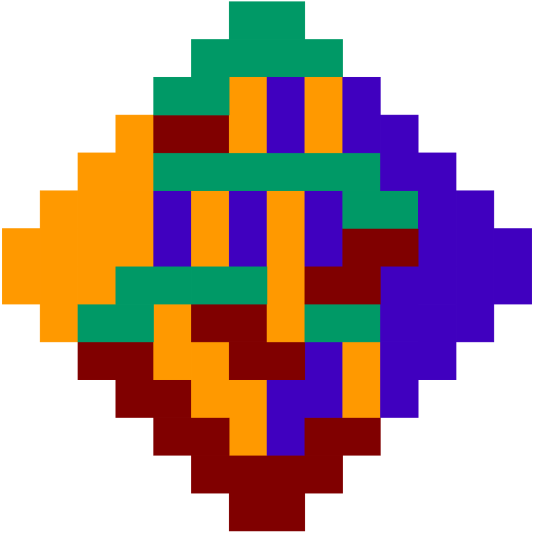
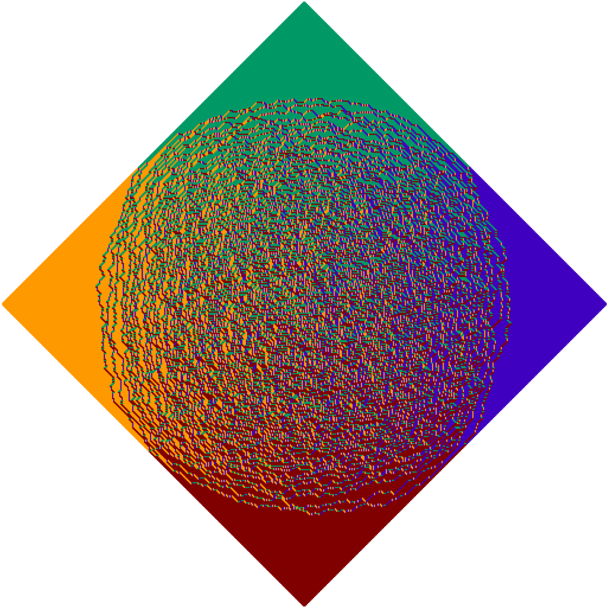
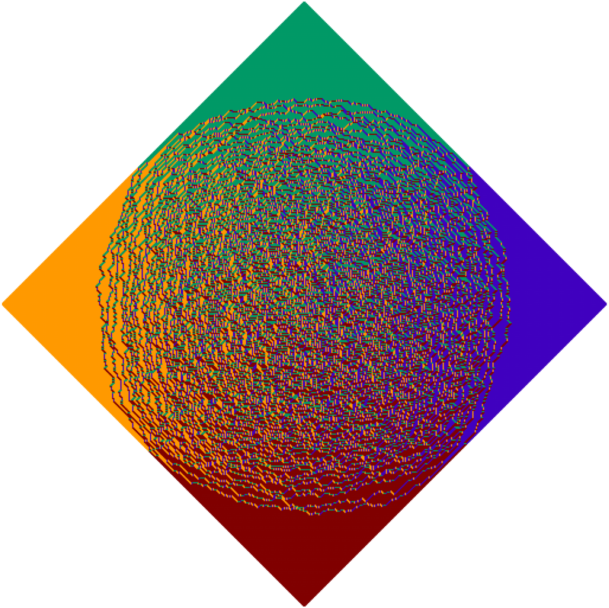
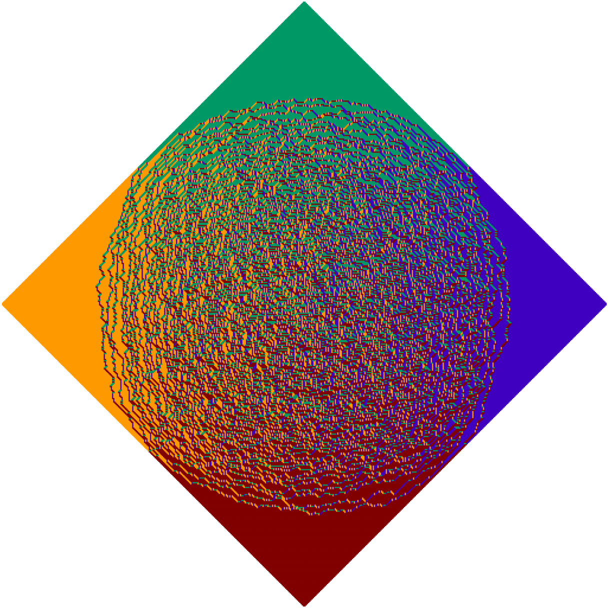




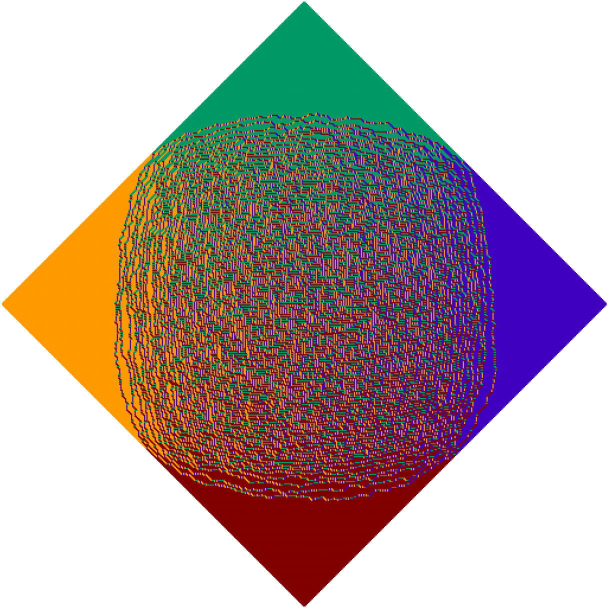
Arctic circle theorem [Jockusch, Propp and Shor 1998]



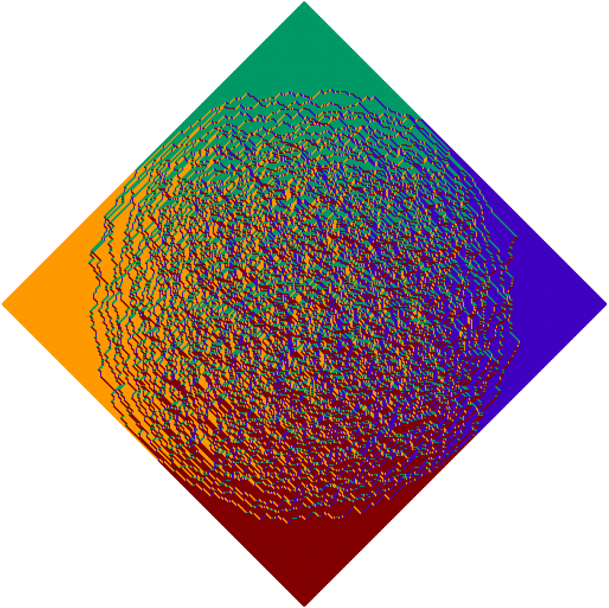
History [Logan, Shepp 1977; Vershik, Kerov 1977; Nienhuis, Hilhorst, Blöte 1982]



Fits into curved CFT formalism [Allegra, Dubail, JMS, Viti 2016]

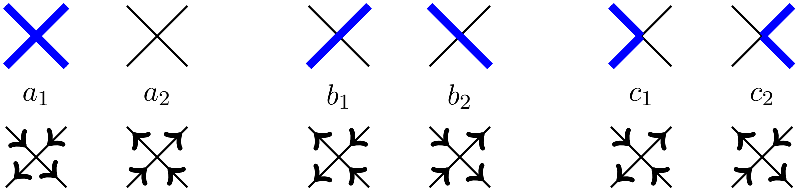


Add attractive interactions between dimers (no theorem)



Add repulsive interactions between dimers (no theorem)

Six-vertex model



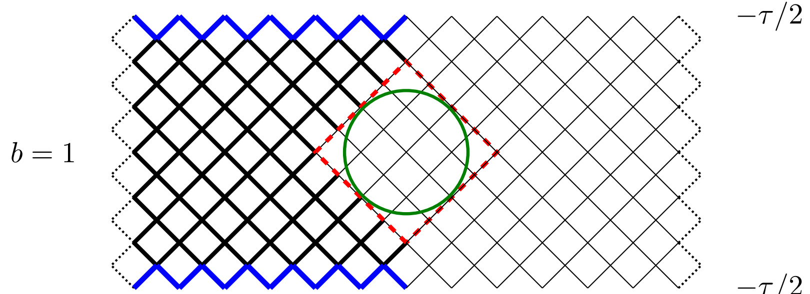
$$a = d\sin(\gamma+\epsilon) \quad , \quad b = d\sin\epsilon \quad , \quad c = d\sin\gamma$$

$$\Delta = \frac{a^2+b^2-c^2}{2ab} = \cos\gamma.$$

Disclaimer: in the following $a = 1$, and Δ is fixed to some value.

An Observation

[JMS 2014] [Allegra, Dubail, JMS, Viti 2016]



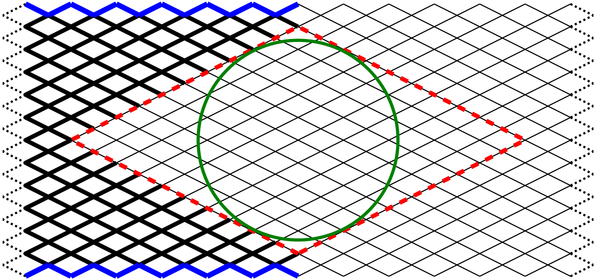
Six vertex model with domain wall boundary conditions

[Korepin 1982]

An Observation

[JMS 2014] [Allegra, Dubail, JMS, Viti 2016]

$b = \frac{1}{2}$



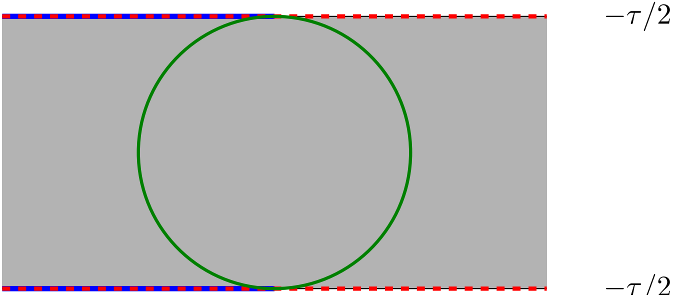
$-\tau/2$

$-\tau/2$

An Observation

[JMS 2014] [Allegra, Dubail, JMS, Viti 2016]

$b \to 0$



Hamiltonian (or Trotter) limit.

Relation through a transfer matrix (6-vertex model)

$$Z_n^{\mathrm{IK}}(b) = \langle \ldots \uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow \ldots | T(b)^{2n} | \ldots \uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow \ldots \rangle$$

partition function of the six vertex model with domain wall boundary conditions.

$$T(b) = 1 + bH_{XXZ} + O(b^2)$$

$$\lim_{N\to\infty} T(\tau/2n)^N = e^{\tau H_{XXZ}}$$

$$\mathcal{Z}(\tau) \;=\; \langle \Psi_0 | e^{\tau H} | \Psi_0 \rangle \;=\; \lim_{n\to\infty} Z_n^{\mathrm{IK}}(b = \frac{\tau}{2n})$$

This is a Trotter limit. Familiar in the context of QTM, QFT, etc.

Izergin-Korepin partition function

There is an exact determinant formula for Z_n^{IK}
[Izergin 1987, Izergin, Coker, Korepin 1992]

In the homogeneous limit it becomes a Hankel determinant:

$$Z_n^{\mathrm{IK}} = \frac{[\sin\epsilon]^{n^2}}{\prod_{k=0}^{n-1} k!^2} \det_{0\leq i,j\leq n-1}\left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u} \frac{1-e^{-\gamma u}}{1-e^{-\pi u}}\right)$$

where recall $b = \frac{\sin\epsilon}{\sin(\gamma+\epsilon)}$ and $\cos\gamma = \Delta$.

Can be rewritten as a Fredholm determinant [Slavnov 2003] (see also [Colomo Pronko 2003])

Hankel matrices and orthogonal polynomials

- Choose a scalar product $\langle f, g\rangle = \int dx f(x)g(x)w(x)$
- Let $\{p_k(x)\}_{k\geq 0}$ be a set of monic orthogonal polynomials for the scalar product , $\langle p_k, p_l\rangle = h_k\delta_{kl}$
- Consider the Hankel matrix A, with elements $A_{ij} = \langle x^{i+j}\rangle$

$$\det A = \prod_{k=0}^{n-1} h_k \quad , \quad (A^{-1})_{ij} = \left.\frac{\partial^{i+j} K_n(x,y)}{i!j!\partial x^i \partial y^j}\right|_{\substack{x=0\\y=0}} \text{ with}$$

$$K_n(x,y) = \sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{h_k} = \frac{1}{h_{n-1}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}$$

The peculiar case of $\Delta = 0$

Orthogonal polynomials may be identified as Meixner-Pollaczek polynomials [Colomo, Pronko 2005], and may be computed explicitly.

$$Z_n(a = 1, b, \Delta = 0) = (1 + b^2)^{n^2/2}$$

$$\mathcal{Z}(\tau) = \lim_{n\to\infty} \left(1 + \frac{\tau^2}{4n^2}\right)^{n^2/2} = e^{\tau^2/8}$$

Relation with several other problems: representations of $U(N)$ [Weyl 1925] and Schur processes, PNG droplet [Prähofer, Spohn 2002], Toeplitz determinants, combinatorics of oscillating Young tableaux [Roby 1995].

Laguerre polynomials

$$w(x) = e^{-\epsilon x} \text{ on } \mathbb{R}_+ \quad , \quad \det(A) = \frac{\prod_{k=0}^{n-1} k!^2}{\epsilon^{n^2}}$$

$$Z_n = \left(\frac{\sin\epsilon}{\epsilon}\right)^{n^2} \times \frac{\det\limits_{0\le i,j\le n-1}\left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u} \dfrac{1-e^{-\gamma u}}{1-e^{-\pi u}}\right)}{\det\limits_{0\le i,j\le n-1}\left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u} \Theta(u)\right)}$$

Now use $\frac{\det A}{\det B} = \det(B^{-1}A) = \det(1 + B^{-1}(A-B))$ to get something well behaved in the Hamiltonian limit.

Result: exact fredholm determinant representation

[JMS 2017]

$$\mathcal{Z}(\tau) \;=\; \langle \Psi_0 | e^{\tau H} | \Psi_0 \rangle \;=\; e^{-\frac{1}{24}(\tau \sin\gamma)^2} \det(I - V)$$

$$V(x,y) = B(x,y)\,\omega(y)$$

$$B(x,y) = \frac{\sqrt{y} J_0(\sqrt{x}) J_0'(\sqrt{y}) - \sqrt{x} J_0(\sqrt{y}) J_0'(\sqrt{x})}{2(x-y)}$$

$$\omega(y) = \Theta(y) - \frac{1 - e^{-\gamma y/(2\tau \sin\gamma)}}{1 - e^{-\pi y/(2\tau \sin\gamma)}}$$

$$\log\det(I - V) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_{\mathbb{R}^k} dx_1 \dots dx_k V(x_1, x_2) \dots V(x_k, x_1)$$

Alternative form using Meixner-Pollaczek polynomials

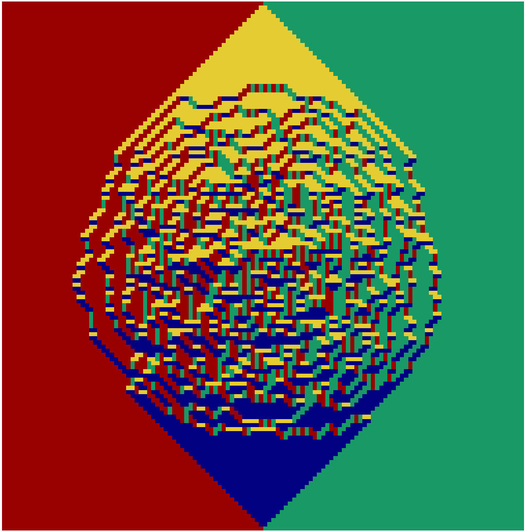
$$\mathcal{Z}(\tau) = e^{-i\frac{\tau}{2}\sin\gamma} \det(I - W)$$

$$W(x,y) = e^{i(\tau\sin\gamma - \gamma)} \frac{f_\tau(x)g_\tau(y) - f_\tau(y)g_\tau(x)}{x-y} \frac{e^{2\gamma y}}{1+e^{2\pi y}}$$

$$f_\tau(x) \; = \; {}_1F_1(ix + 1/2, 1, -i\tau\sin\gamma) \qquad , \qquad g_\tau(x) = -\tau\partial_\tau f_\tau(x)$$

$${}_1F_1(a,b,z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!} \quad , \quad (c)_k = c(c{+}1)\ldots(c{+}k{-}1) = \frac{\Gamma(c+k)}{\Gamma(c)}$$

With pictures



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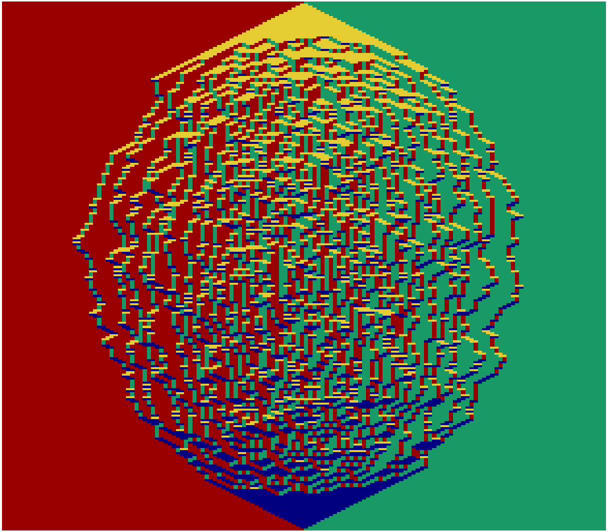

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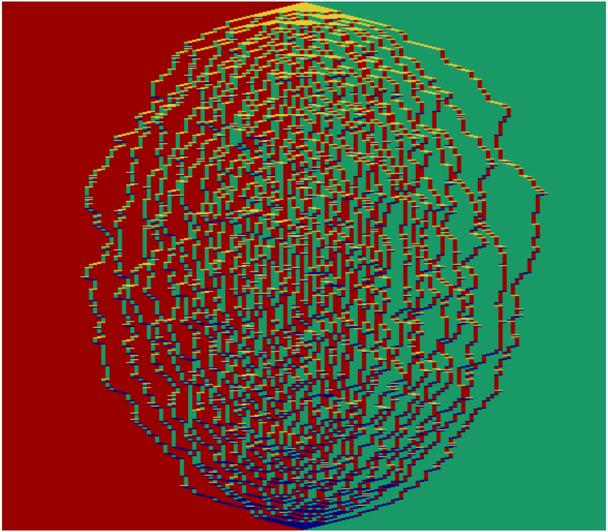
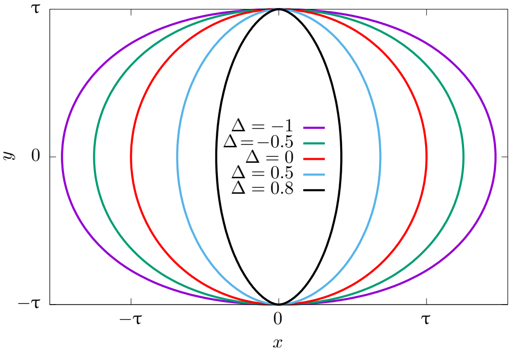


Area law and arctic curves ($\alpha = \pi/(\pi-\gamma)$)

$$\frac{x(s)}{\tau} = \frac{\alpha^2 \csc^2 \alpha s\left\{\cos(2\gamma+3s)(\cos s - \alpha \sin s \cot \alpha s) + \alpha \sin s \cos s \cot \alpha s + \cos^2 s - 2\right\} + 2}{\csc s \csc(\gamma+s)(\sin^2(\gamma+s) + \sin^2 s)}$$

$$\frac{y(s)}{\tau} = \frac{\left[2\alpha^2 \csc \gamma \sin^2 s \csc^2 \alpha s\{2\alpha \sin s \cot \alpha s \sin(\gamma+s) - \sin(\gamma+2s)\} - 1\right] + \sin^2 s}{\csc^2(\gamma+s)(\sin^2(\gamma+s) + \sin^2 s)}$$



[Colomo, Pronko 2009]

Asymptotics

Easiest: use [Zinn-Justin 2000] [Bleher, Fokin 2006]

$$\mathcal{Z}(\tau) \underset{\tau\to\infty}{\sim} \exp\left(\left[\frac{\pi^2}{(\pi-\gamma)^2}-1\right]\frac{(\tau\sin\gamma)^2}{24}\right)\tau^{\kappa(\gamma)}O(1)$$

$$\kappa(\gamma) = \frac{1}{12} - \frac{(\pi-\gamma)^2}{6\pi\gamma}$$

Interpretation: free energy of the fluctuating region.

Back to real time

Analytic continuation

- Return probability: $\tau = it$
- Correlations: $y = it$ and $\tau \to 0^+$

Continuation of the arctic curves should give the light cone:

Free fermions: $x^2 + y^2 = (\tau/2)^2 \quad \longrightarrow \quad x = \pm t$
Interactions: $\text{complicated} \quad \longrightarrow \quad x = \pm(\sin\gamma)t = \pm\sqrt{1-\Delta^2}\, t$

This coincides exactly with the result of generalized hydrodynamics

Analytic continuation

Numerical observations (huge precision, t up to 600 on laptop):

- Root of unity, $\gamma = \arccos \Delta = \frac{\pi p}{q}$

$$-\log \mathcal{R}(t) = \left(\frac{q^2}{(q-1)^2} - 1 \right) \frac{(t \sin \gamma)^2}{12} + O(\log t)$$

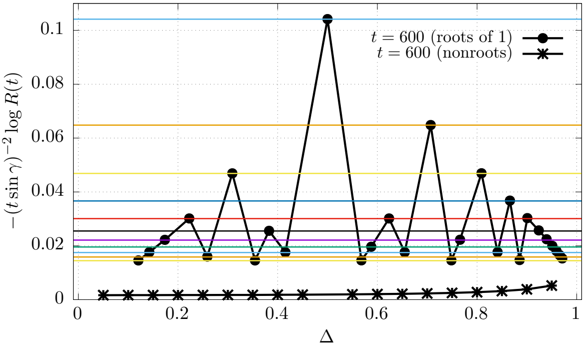
  Coincides with analytic continuation only when $p = 1$.

- non root of unity

$$-\log \mathcal{R}(t) = t \sin \gamma + O(\log t)$$

Analytic continuation

Numerical observations (huge precision, t up to 600 on laptop):



Fractal behavior also compatible also with [De Luca, Collura, Viti 2017]

How about a proof using Riemann-Hilbert techniques?

[Its, Izergin, Korepin, Slavnov 1990]

The special case $\Delta = 1$

$\mathcal{R}(t) = |\det(I - K)|^2$ on $L^2([0;\sqrt{t}])$.

$$K(u,v) = i\sqrt{uv}J_0(uv)e^{-\frac{1}{2}i\left(u^2+v^2\right)} \quad \longrightarrow \quad \frac{e^{i\pi/4}}{\sqrt{2\pi}}e^{-\frac{i}{2}(u-v)^2}$$

Nb: in imaginary time, this coincides with an exact large deviation result in SSEP [Derrida, Gerschenfeld 2009]

Computing each $\operatorname{Tr} K^n \propto \sqrt{t}$ asymptotically is doable. In the end:

$$\mathcal{R}(t) \sim \exp\left(-\zeta(3/2)\sqrt{t/\pi}\right) t^{1/2} O(1)$$

By the previous logic ($x(t) \sim t^\delta \Rightarrow \mathcal{R}(t) \lesssim e^{-at^\delta}$), we find $\delta \leq 1/2$. Transport is at most diffusive for this quench, probably diffusive.

The emptiness formation probability

$$E(x,t) = \langle \Psi(t) | P_x | \Psi(t) \rangle \qquad , \qquad P_x = \prod_{j=-\infty}^{x} \left(\frac{1+\sigma_j^z}{2} \right)$$

Probability that all spins on the left of $j = x$ are up at time t.

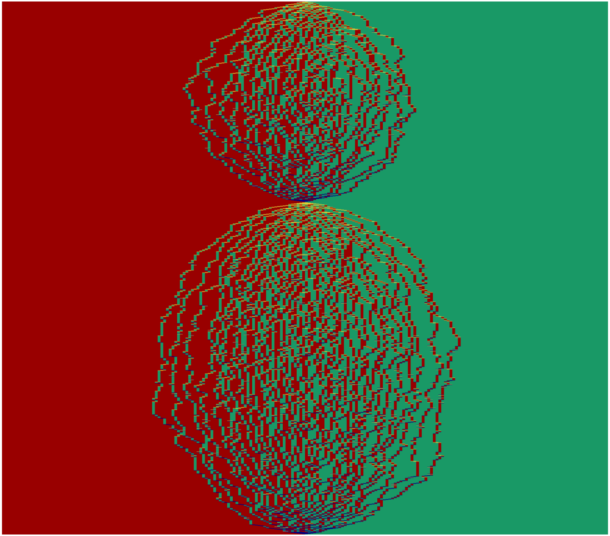
Simple cases (last two follow from conservation of magnetization):

$$E(x \ll -t, t) = 1$$

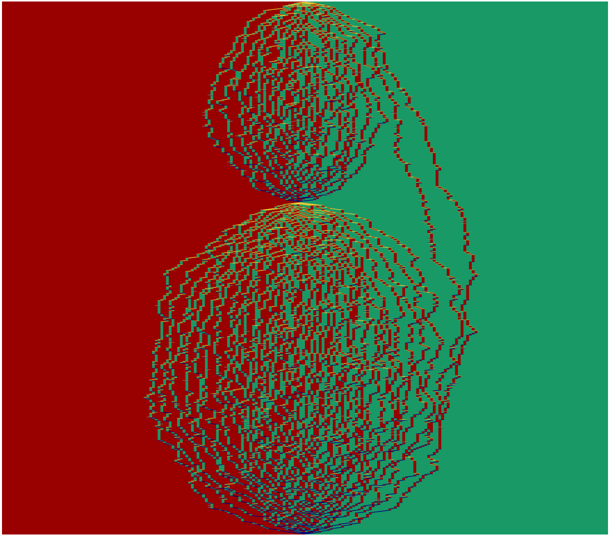
$$E(x > 0, t) = 0$$

$$E(x = 0, t) = |\langle \psi(t) | \psi_0 \rangle|^2$$

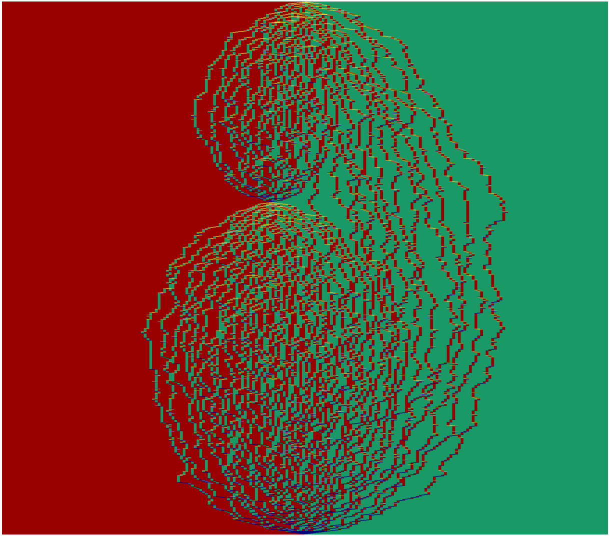
With pictures



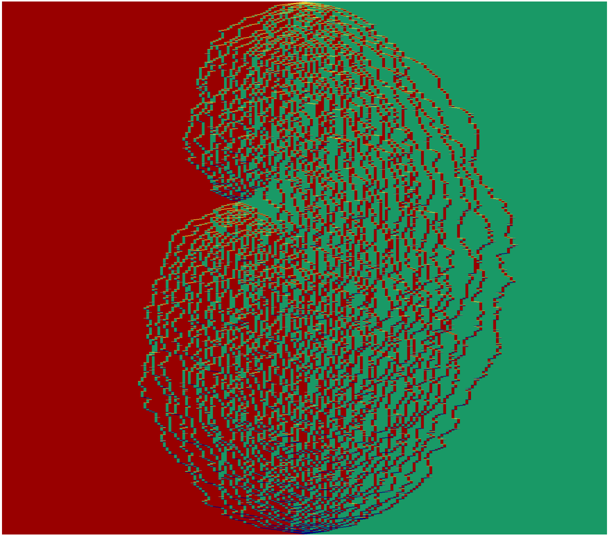
With pictures



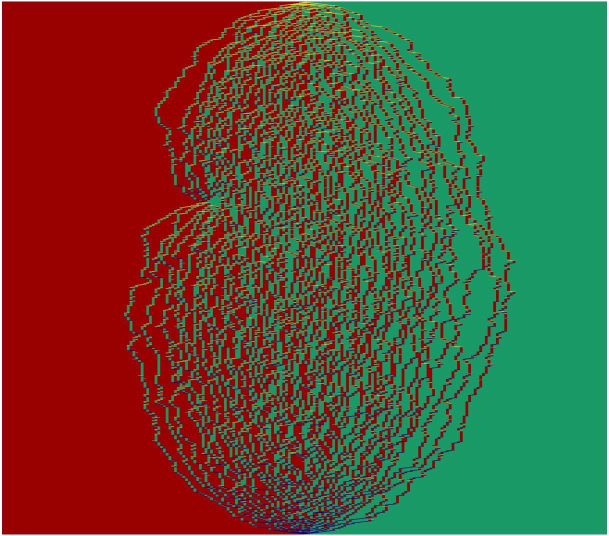
With pictures



With pictures



With pictures



Exact formula for the emptiness

$$a = \sin(\gamma + \epsilon), \quad b = \sin(\epsilon), \quad \Delta = \cos\gamma, \quad \varphi(\epsilon) = \frac{\sin\gamma}{\sin(\gamma+\epsilon)\sin\epsilon}$$

$$E_N(a, b, \Delta | r, s) = \frac{\det\limits_{1\le j,k\le N} \begin{bmatrix} \partial\xi_k^{j-1} & (k \le s) \\ \partial_\epsilon^{j+k-s-2}\varphi(\epsilon) & (k > s) \end{bmatrix}}{\det\limits_{1\le j,k\le N} \left[\partial_\epsilon^{j+k-2}\varphi(\epsilon)\right]} F(\xi_1, \ldots, \xi_s)\Big|_{\xi_1=\ldots=\xi_s=0}$$

$$F(\{\xi_i\}) = \prod_{j=1}^{s} \frac{(N-j)![\sin\xi_j]^{N-r}\sin(\xi_j-\gamma)^r}{\sin(\gamma+\epsilon)^r[\sin\epsilon]^{N-r}\sin(\xi_j+\epsilon)^N} \prod_{1\le j<k\le s} \frac{\sin(\xi_j+\gamma+\epsilon)\sin(\xi_k+\epsilon)}{\sin(\xi_j-\xi_k+\gamma)}$$

[Colomo & Pronko 2007], then take the Trotter limit. [To do]

Example of free fermions (ballistic)

- Exact determinant formula $E(x,t) = \det(\ldots)$

- Asymptotics:

$$E(x,t) \sim e^{-t^2 f(x/t)}$$

with $f(s) = \frac{1}{4} + s + \frac{s^2}{4}(3 - 2\log(-s))$

Conjectures away from free fermions

- Root of unity:

$$E(x,t) \sim e^{-t^2 f_\gamma(x/t)}$$

where f_γ is nowhere continuous behavior as a function of γ.

- Heisenberg point $(\Delta = 1)$

$$E(x,t) \sim \exp\left[-\sqrt{t} f(x/\sqrt{t})\right] t^{\cdots} O(1)$$

which would mean diffusive transport.

Conclusion

- Exact computations, valid for all x and t.
- Large x, t behavior is compatible with GHD.
- Relation with arctic circle problems.
- For $\Delta = 1$, results do suggest diffusive behavior.

Thank you!