Analytic Calculation of the Conformal Partition Functions of Critical Bond Percolation on the Square Lattice

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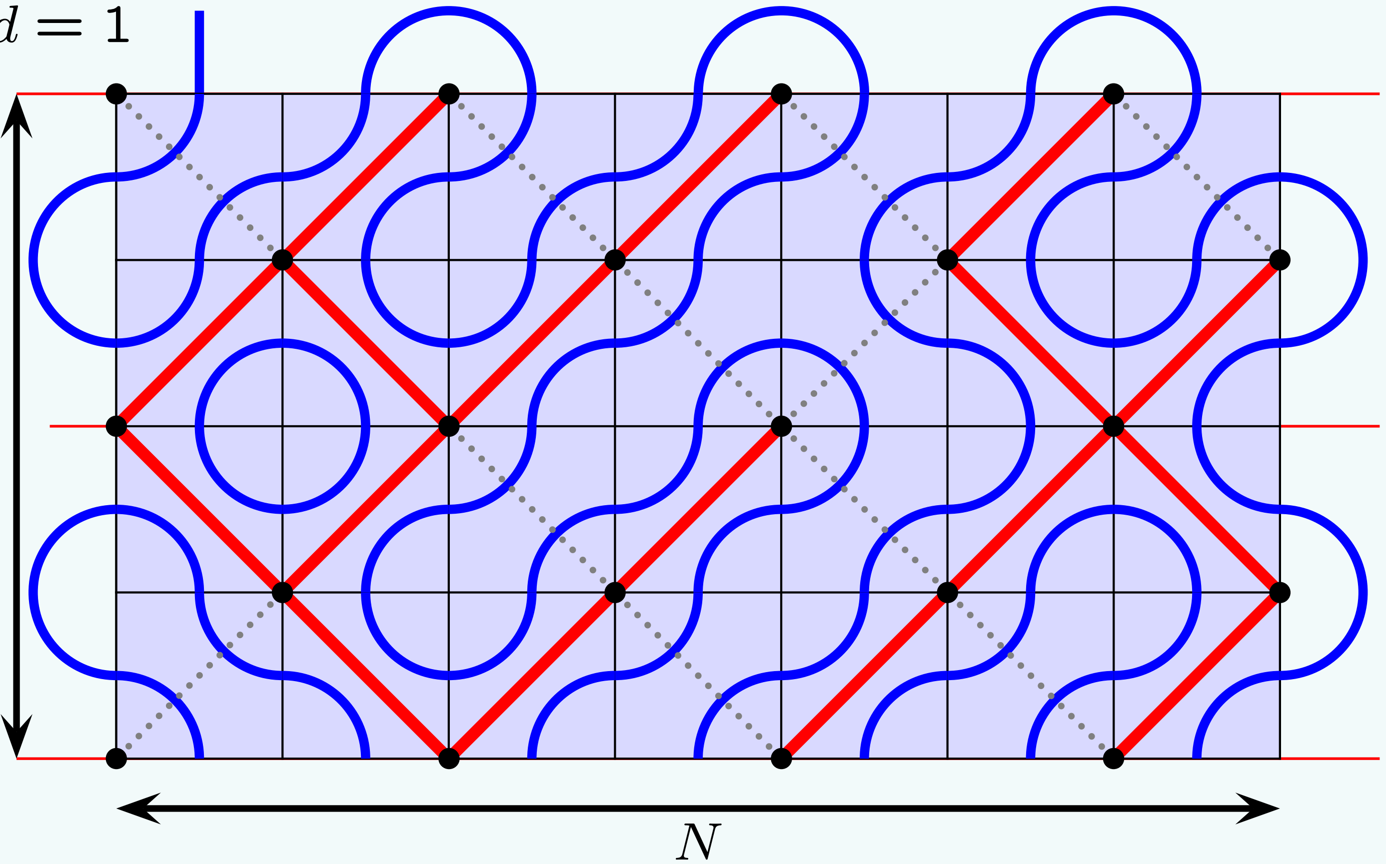

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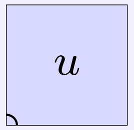
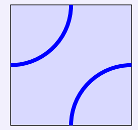
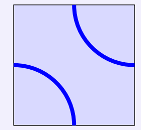
Part 1

Logarithmic Minimal Models and Conformal Description of Percolation

Logarithmic Minimal Models $\mathcal{LM}(p,p')$

- In the Temperley-Lieb algebra, the critical face operators (PRZ2006) are

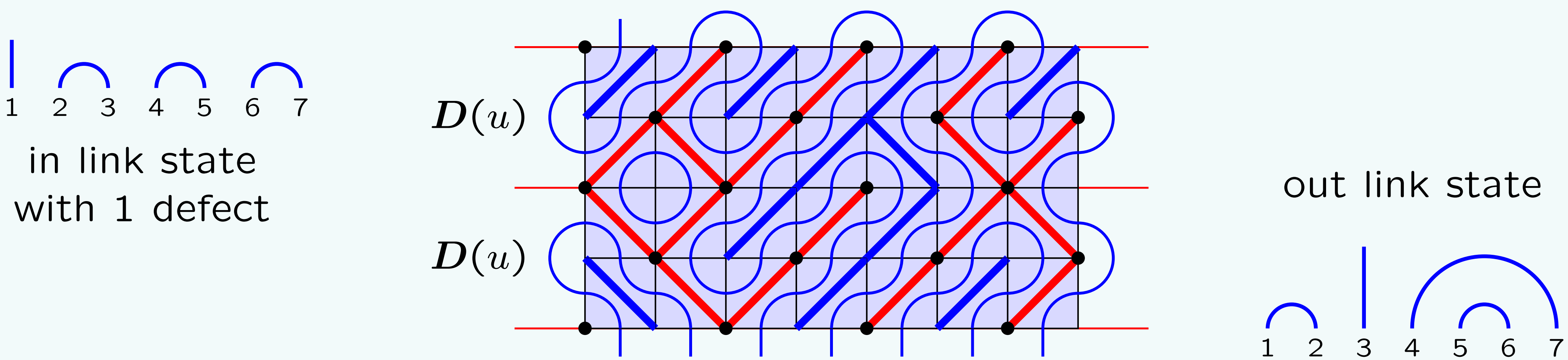
$X(u) =$  $= \sin(\lambda-u)$  $+ \sin u$  ; $\qquad X_j(u) = \sin(\lambda-u)\, I + \sin u\, e_j$

$$1 \le p < p' \text{ coprime integers}, \qquad \lambda = \frac{(p'-p)\pi}{p'} = \text{crossing parameter}$$

$$u = \text{spectral parameter}, \qquad \beta = 2\cos\lambda = \text{loop fugacity}$$

$$\boxed{Z = \sum_{\text{loop configs}} \sin^{N_1}(\lambda-u)\sin^{N_2}u\ \beta^{\#\text{ loops}}}$$

- There are no local degrees of freedom only extended (nonlocal) loop segments (connectivities).
- $\mathcal{LM}(2,3)$ **is Critical Percolation:** $\quad (p,p')=(2,3), \quad \lambda=\frac{\pi}{3}, \quad u = \frac{\lambda}{2} = \frac{\pi}{6}$ (isotropic)



Kesten 1980: Critical (isotropic) bond probability $= p_c = \sin(\lambda-u) = \sin u = \frac{1}{2}$

Duplantier 1988: Critical bond percolation (red bonds) $\Leftrightarrow$ Loop (hull) percolation

$\beta = 1 \qquad \Rightarrow \qquad$ stochastic process

Critical Percolation $\mathcal{LM}(2,3)$ as a Log CFT

- In the continuum scaling limit, critical percolation yields a logarithmic CFT:
- **Central charge:** $(p,p') = (2,3)$

$$\boxed{c \;=\; 1 - \frac{6(p-p')^2}{pp'} \;=\; 0}$$

- **Infinitely extended Kac table of conformal weights:**

$$\boxed{\begin{aligned}\Delta_{r,s} &= \frac{(p'r-ps)^2-(p-p')^2}{4pp'}\\ &= \frac{(3r-2s)^2-1}{24}, \qquad r,s=1,2,3,\ldots\end{aligned}}$$

More generally, $r,s \in \frac{1}{2}\mathbb{N}_{\geq 0}$.

- **Kac characters:**

$$\boxed{\chi_{r,s}(q) = q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}}$$

- **Quantum number:** s is given by

$$\boxed{s = \begin{cases} d+1, & \text{strip} \\ \frac{d}{2}, & \text{cylinder}\end{cases} \qquad d = \#\ \text{defects}}$$

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Part 2

Integrability, Quantum Chains and Transfer Tangles

Yang-Baxter Integrability of Temperley-Lieb Models

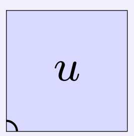
- Root of Unity Crossing Parameter

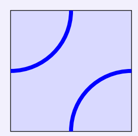
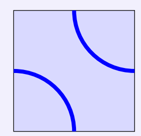
$$\lambda = \frac{(p'-p)\pi}{p'}, \qquad 1 \le p < p', \qquad (p,p') \text{ coprime integers}$$

- The Temperley-Lieb (TL) algebra is generated by the identity I and $\{e_j\}_{j=1}^{L}$ with relations

$$e_j^2 = \beta\, e_j, \qquad e_j e_{j\pm1} e_j = e_j, \qquad [e_j, e_k] = 0, \quad |j-k|>1, \qquad \beta = 2\cos\lambda$$

- Local Face Transfer Operators

$X(u)\ =$ 

$=\ \sin(\lambda-u)$  $+\ \sin u$  ; $\qquad X_j(u)\ =\ \sin(\lambda-u)\, I + \sin u\, e_j$

- The face operators automatically satisfy the Local Inversion and Yang-Baxter Equations

$$X_j(u)X_j(-u) = \sin(\lambda-u)\sin(\lambda+u)\, I$$

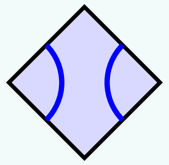
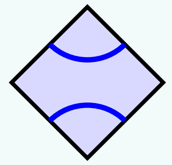
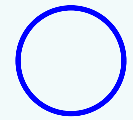
$$X_j(u)X_{j+1}(u+v)X_j(v) = X_{j+1}(v)X_j(u+v)X_{j+1}(u)$$

- Integrability

$$\text{Inv} + \text{YBE} \ \Rightarrow\ \text{Commuting Transfer Matrices} \ \Rightarrow\ \text{Integrable}$$

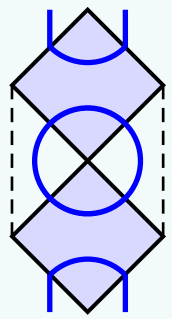
Diagrammatic TL Algebra on the Strip

- The TL algebra admits a planar diagrammatic representation consisting of "monoid" diagrams. Physically this is a loop gas, mathematically it is the loop representation

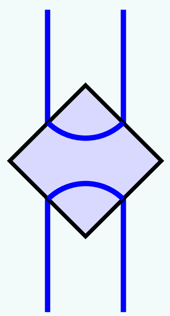




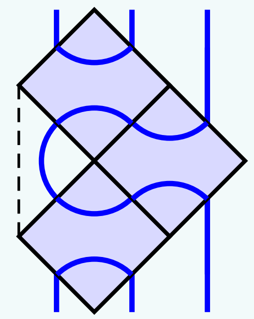
$= \beta = 2\cos\lambda =$ loop fugacity

- Multiplication is by juxtaposition. The monoids satisfy



$= \beta$





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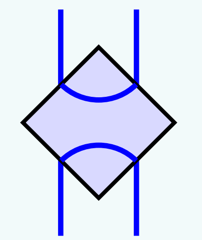


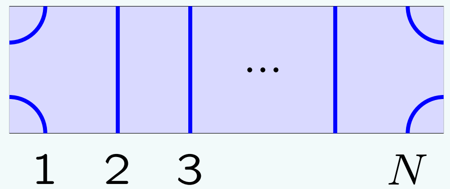
- The quantum Hamiltonian associated with the commuting double row transfer matrix $\boldsymbol{D}(u)$ is

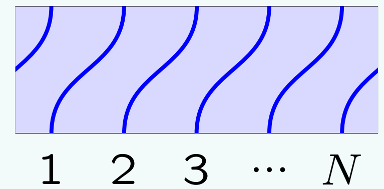
$$\boxed{-H = \frac{d}{du}\log \boldsymbol{D}(u)\Big|_{u=0} = \sum_{j=1}^{N-1}(e_j - I) = \text{quantum chain}}$$

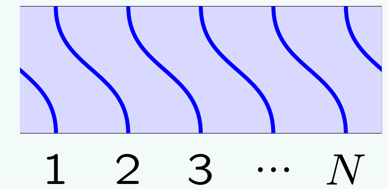
This precisely coincides with the Markov matrix of the Raise and Peel model (GNPR2004) which is a conformal stochastic growth process of a fluctuating 1-d interface.

Enlarged Periodic TL Algebra on the Cylinder

- On the cylinder, we add an extra monoid and the shift operators

$e_N =$ 

, $\Omega =$ 

, $\Omega^{-1} =$ 

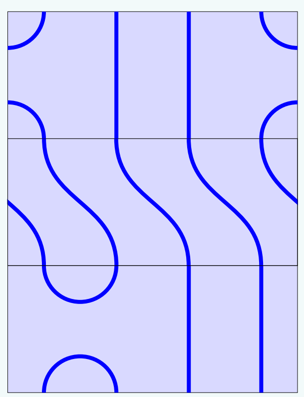
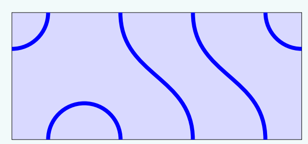
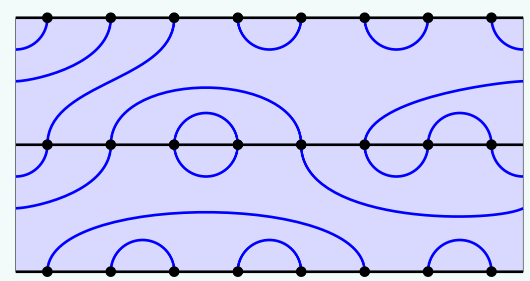
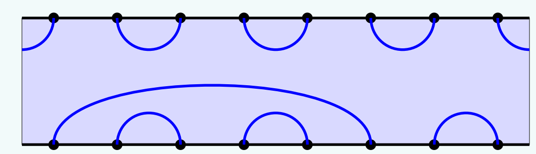
- Taking j mod N with N even leads to the extra relations

$$\Omega\Omega^{-1} = \Omega^{-1}\Omega = I, \quad \Omega e_j \Omega^{-1} = e_{j-1}, \quad \Omega^N e_N = e_N \Omega^N, \quad (\Omega^{\pm 1} e_N)^{N-1} = \Omega^{\pm N}(\Omega^{\pm 1} e_N)$$

$$E\Omega^{\pm 1}E = \alpha E \qquad \text{where} \qquad E = e_2 e_4 \ldots e_{N-2} e_N$$

For $d = 0$, the last relation removes any non-contractible loops and assigns them a fugacity α.

- For example

$e_1\Omega^{-1}e_4 =$  $= \beta$  ,  $= \alpha\beta$ 

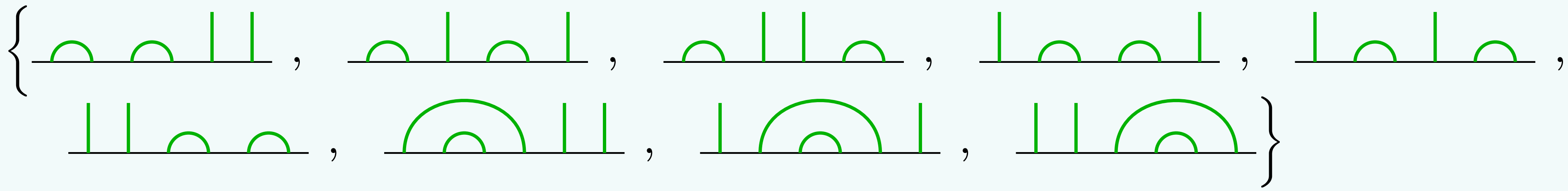
- The quantum Hamiltonian associated with the commuting single row transfer matrix $\boldsymbol{T}(u)$ is

$$-H = \frac{d}{du} \log \boldsymbol{T}(u)\Big|_{u=0} = \sum_{j=1}^{N} (e_j - I) = \text{quantum chain}$$

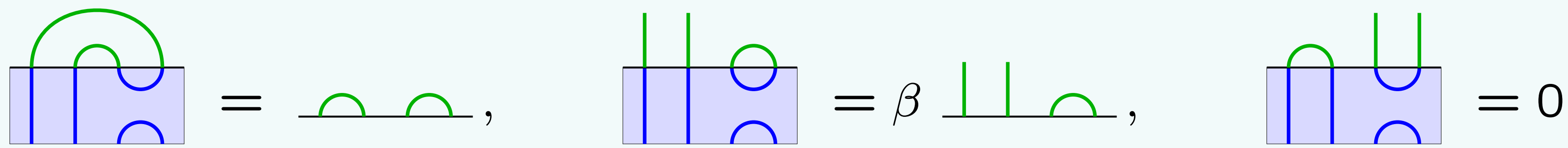
Link States

- Link states on the strip (standard module V_N^d): $\quad \dim V_N^d = \binom{N}{\frac{N-d}{2}} - \binom{N}{\frac{N-d-2}{2}}$

$V_{N=6}^{d=2} =$

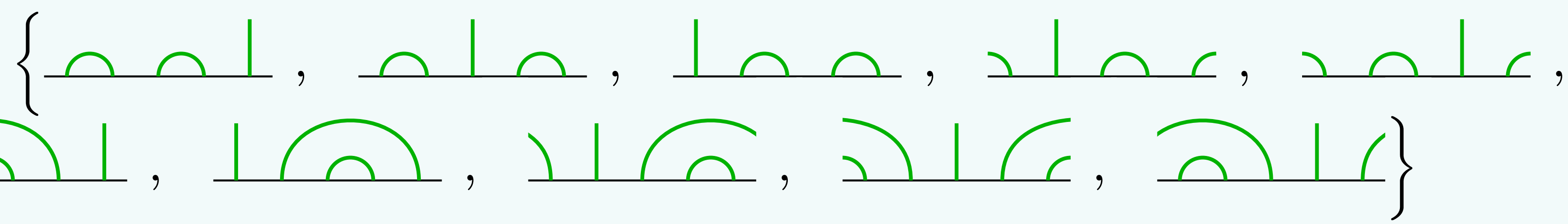


The (defect preserving) action of the TL algebra on the strip is

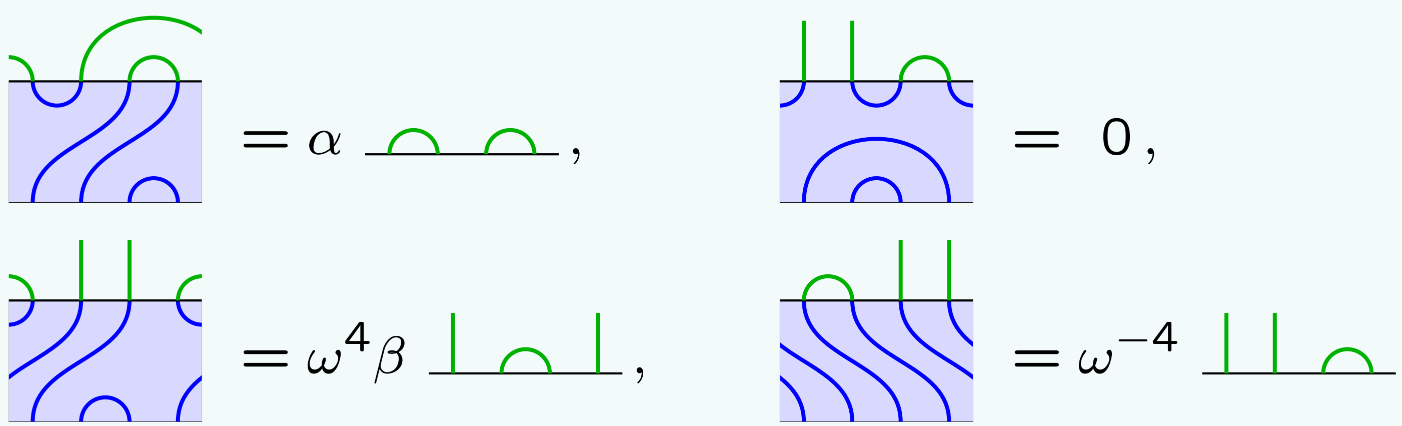


- Link states on the cylinder (standard module W_N^d): $\quad \dim W_N^d = \binom{N}{\frac{N-d}{2}}$

$W_{N=5}^{d=1} =$



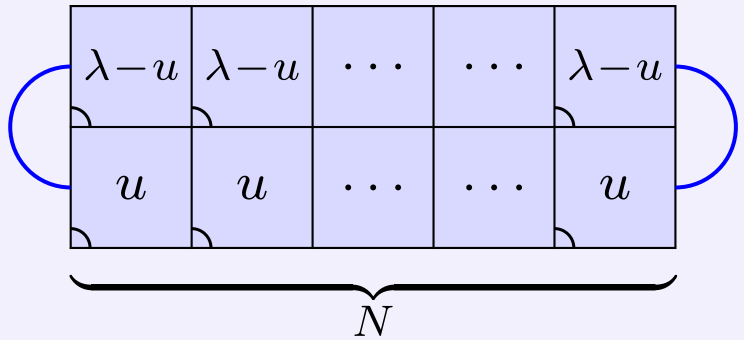
The (defect preserving) action of the enlarged periodic TL algebra on the cylinder is

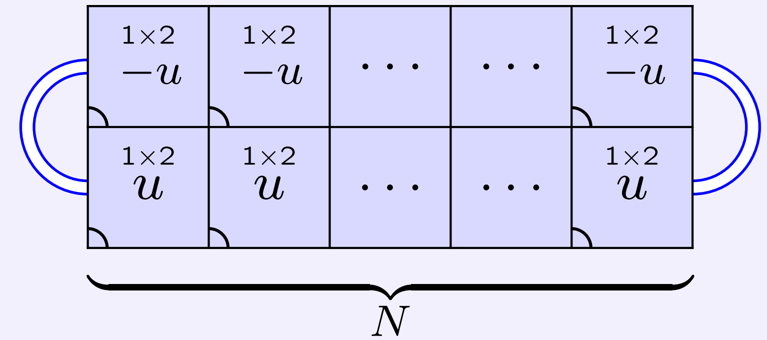


We set $\alpha = 2\cos\gamma = \{$non-contractible loop fugacity$\}$, $\omega = e^{i\gamma} = \{$winding parameter$\}$
Ultimately, we let $\gamma \to 0$ so that $\omega = 1$ and $\alpha = 2$.

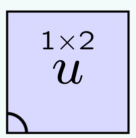
Percolation Transfer Tangles

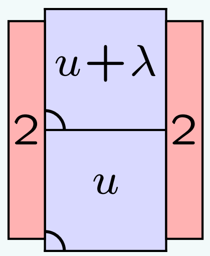
- Setting $\boldsymbol{D}^1(u) = \boldsymbol{D}(u)$ on the strip, the 1×1 and 1×2 fused double-row transfer tangles are

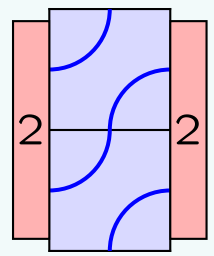
$\boldsymbol{D}^1(u) = (-1)^N$ 

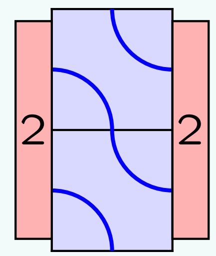
$\boldsymbol{D}^2(u) =$ 

where the 1×2 fused face operator is



$= \dfrac{1}{s_0(u)}$ 

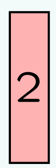
$= s_1(-u)$ 

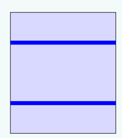
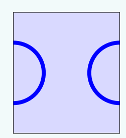
$+\, s_1(u)$ 

,

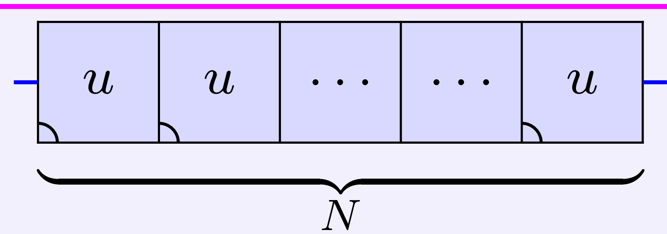
$$s_k(u) = \frac{\sin(u+k\lambda)}{\sin\lambda}$$

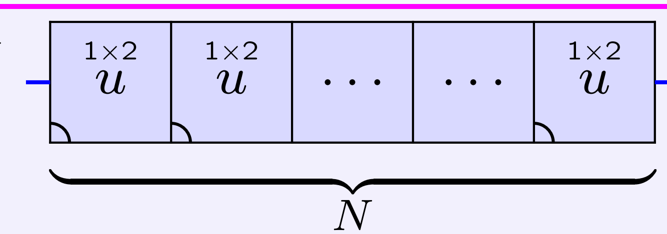
The projector is



$=$  $-\ \beta^{-1}$  , $\beta = 1$ for percolation

- Setting $\boldsymbol{T}^1(u) = \boldsymbol{T}(u)$, the 1×1 and 1×2 fused periodic single row transfer matrices are

$\boldsymbol{T}^1(u) = i^N$ 

$\boldsymbol{T}^2(u) = (-1)^N$ 

- For $\alpha = 2$, a modified trace (MDPR2013) is required to glue the cylinder to form a torus

$$Z^N = \mathrm{tr}\,\boldsymbol{T}(u)^M = \sum_{d\geq 0}(2-\delta_{d,0})Z_d^N = \sum_{d\geq 0}(2-\delta_{d,0})\,\mathrm{Tr}\,\boldsymbol{T}(u)^M\Big|_{\mathsf{W}_{N,\omega=1}^d} = \text{torus partition function}$$

Part 3

T-Systems, Y-Systems, TBA and Physical Combinatorics

Closed Fusion Hierarchies for $\mathcal{LM}(p,p')$

- Let $\boldsymbol{T}_0^1 = \boldsymbol{T}(u)$ be the fundamental periodic transfer matrix. For $n,k \in \mathbb{Z}$, the fused transfer matrices are given recursively by the fusion hierarchies (BazhResh89,M-DuchesnePRasm2014)

$$\boxed{\begin{aligned} \boldsymbol{T}_0^n \boldsymbol{T}_n^1 &= f_n \boldsymbol{T}_0^{n-1} + f_{n-1} \boldsymbol{T}_0^{n+1} \\ \boldsymbol{T}_0^1 \boldsymbol{T}_1^n &= f_{-1} \boldsymbol{T}_2^{n-1} + f_0 \boldsymbol{T}_0^{n+1} \end{aligned}}$$

with

$$\boxed{\begin{array}{c} \boldsymbol{T}_k^n = \boldsymbol{T}^n(u+k\lambda), \qquad f_k = f(u+k\lambda) = s_k(u)^N = \Big(\dfrac{\sin(u+k\lambda)}{\sin\lambda}\Big)^N \\ \boldsymbol{T}_0^{-1} = 0, \qquad \boldsymbol{T}_0^0 = f_{-1}\boldsymbol{I}, \qquad \boldsymbol{T}_0^{-n} = -\boldsymbol{T}_{-n+1}^{n-2} \end{array}}$$

- The closure/periodicity relations are

$$\boxed{\boldsymbol{T}_0^{p'} = \boldsymbol{T}_1^{p'-2} + 2\sigma \boldsymbol{J}\,\boldsymbol{T}_0^0, \qquad \boldsymbol{T}_{p'}^n = \sigma^2 \boldsymbol{T}_0^n, \qquad f_{p'} = \sigma^2 f_0}$$

where $\sigma = i^{-N(p'-p)}$. The matrix $\boldsymbol{J}$ is related to the braid limit $(u \to i\infty)$ and has eigenvalues $J_d = \frac{1}{2}(\omega^{p'} i^{-pd} + \omega^{-p'} i^{pd})$ where the winding parameter ω and defect number d label the standard representations.

- Similar equations hold on the strip with additional $O(1)$ terms.

T- and Y-Systems

- The periodic transfer matrices satisfy the T- and Y-systems (KlümperP92,M-DPR2014)

$$\boxed{\begin{array}{lll} \boldsymbol{T}_0^n \boldsymbol{T}_1^n = f_{-1} f_n \boldsymbol{I} + \boldsymbol{T}_0^{n+1} \boldsymbol{T}_1^{n-1}, & n \geq 0 & \\ & & \boldsymbol{t}_0^n = \dfrac{\boldsymbol{T}_1^{n-1}\boldsymbol{T}_0^{n+1}}{f_{-1} f_n} \\ \boldsymbol{t}_0^n \boldsymbol{t}_1^n = (\boldsymbol{I} + \boldsymbol{t}_1^{n-1})(\boldsymbol{I} + \boldsymbol{t}_0^{n+1}), & n = 1, 2, \ldots, p'-2 & \end{array}}$$

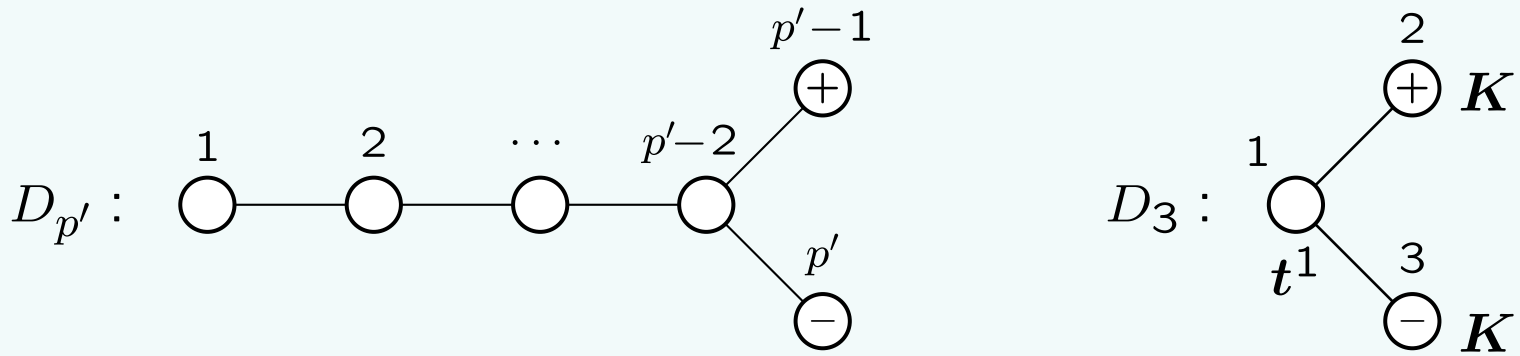
- The Y-systems close for $\mathcal{LM}(p,p')$ with the closure relations (M-DuchesneKlümperP2017)

$$\boxed{\boldsymbol{I} + \boldsymbol{t}_0^{p'-1} = (\boldsymbol{I} + e^{i\boldsymbol{\Lambda}}\boldsymbol{K}_0)(\boldsymbol{I} + e^{-i\boldsymbol{\Lambda}}\boldsymbol{K}_0), \qquad \boldsymbol{K}_0 \boldsymbol{K}_1 = 1 + \boldsymbol{t}_1^{p'-2}}$$

where

$$\boldsymbol{K}_0 = \frac{i^{N(p'-p)}}{f_{-1}} \boldsymbol{T}_1^{p'-2}, \qquad \boldsymbol{J} = \cos\boldsymbol{\Lambda} = T_{p'}(\tfrac{1}{2}\boldsymbol{T}(i\infty)) = \text{Chebyshev polynomial}$$

- The TBA Dynkin diagram (with endpoint nodes distinguished by factors $e^{\pm i\boldsymbol{\Lambda}}$) is



- For bond percolation $\mathcal{LM}(2,3)$, the Y-system is

$$\boxed{\boldsymbol{t}_0^1 \boldsymbol{t}_1^1 = (\boldsymbol{I} + e^{i\boldsymbol{\Lambda}}\boldsymbol{K}_0)(\boldsymbol{I} + e^{-i\boldsymbol{\Lambda}}\boldsymbol{K}_0), \qquad \boldsymbol{K}_0 \boldsymbol{K}_1 = \boldsymbol{I} + \boldsymbol{t}_1^1}$$

- The same Y-system holds for double row transfer matrices on the strip with $\boldsymbol{\Lambda} = 0$.

The $O(1)$ terms cancel out of the universal Y-system.

Percolation: Ground State TBA

- The universal Y-system for percolation on the cylinder takes the form

$$\boxed{\begin{aligned} \mathfrak{a}^1(x-\tfrac{i\pi}{2})\mathfrak{a}^1(x+\tfrac{i\pi}{2}) &= [\boldsymbol{I}+e^{i\boldsymbol{\Lambda}}\mathfrak{a}^2(x)][\boldsymbol{I}+e^{-i\boldsymbol{\Lambda}}\mathfrak{a}^2(x)] \\ \mathfrak{a}^2(x-\tfrac{i\pi}{2})\mathfrak{a}^2(x+\tfrac{i\pi}{2}) &= \boldsymbol{I}+\mathfrak{a}^1(x) \end{aligned}}$$

$$\mathfrak{a}^1(x)=\frac{h_0}{h_1h_{-1}}\boldsymbol{T}^2(u)\Big|_{u=\frac{ix}{3}},\quad \mathfrak{a}^2(x)=\frac{i^N}{h_{-2}}\boldsymbol{T}^1(u)\Big|_{u=\frac{ix}{3}+\frac{\pi}{3}},\quad \boldsymbol{J}=\cos\boldsymbol{\Lambda},\quad h_k=s_k(u)^N$$

- Using Fourier transforms, these lead to nonlinear integral equations in the form of Thermodynamic Bethe Ansatz (TBA) equations. For the ground state in the sector with $d=0$ defects

$$\boxed{\begin{aligned} \log\mathfrak{a}^1(x) &= \log\tanh^N\tfrac{x}{2}+K*\log[1+\mathfrak{a}^2(x)]^2 \\ \log\mathfrak{a}^2(x) &= K*\log[1+\mathfrak{a}^1(x)] \end{aligned}}$$

where

$$K(x)=\frac{1}{2\pi\cosh x},\qquad (f*g)(x)=\int_{-\infty}^{\infty}dy\, f(x-y)\,g(y)$$

- Using the dilogarithm techniques of KlümperP92 gives the expected finite-size corrections

$$\boxed{\begin{gathered} -\log T^1(u)=Nf_{\text{bulk}}(u)+\tfrac{2\pi i}{N}\Big[e^{-3iu}(-\tfrac{c}{24}+\Delta)-e^{3iu}(-\tfrac{c}{24}+\bar\Delta)\Big]+o(\tfrac{1}{N}) \\ c=0 \qquad \Delta=\bar\Delta=-\tfrac{1}{24} \end{gathered}}$$

- More generally, for the ground state in the sector with d defects

$$\boxed{\Delta=\bar\Delta=\Delta_{0,\frac{d}{2}}=\tfrac{d^2-1}{24}=-\tfrac{1}{24},\,0,\,\tfrac{1}{8},\,\tfrac{1}{3},\,\tfrac{5}{8},\,1,\,\tfrac{35}{24},\,2,\,\tfrac{21}{8},\,\tfrac{10}{3},\,\tfrac{33}{8},\,5,\ldots}$$

- Similar equations hold on the strip with 1 copy of Virasoro (complex conjugation symmetry) and $\boldsymbol{\Lambda}=0$.

Dilogarithm Identities

- Analytic expressions for the conformal weights $\Delta_{1,s}$, $s \in \mathbb{N}_{\geq 0}$, involve the dilogarithm integrals

$$\boxed{\mathcal{K}_\sigma(\gamma) = \mathcal{I}_1(\gamma) + \mathcal{I}_2(\gamma) + \mathcal{I}_3(\gamma), \qquad \sigma = \pm 1, \quad \gamma = 0, \frac{2\pi}{3}}$$

where

$$\mathcal{I}_1(\gamma) = \int_0^{\mathrm{a}^1(\infty)} d\mathrm{a} \left(\frac{\log(1+\mathrm{a})}{\mathrm{a}} - \frac{\log|\mathrm{a}|}{1+\mathrm{a}} \right), \qquad \mathrm{a}^1(\infty) = 4\cos^2\gamma - 1$$

$$\mathcal{I}_2(\gamma) = \int_{\mathrm{a}^2(-\infty)}^{\mathrm{a}^2(\infty)} d\mathrm{a} \left(\frac{\log\left(1 + e^{3i\gamma}\,\mathrm{a}\right)}{\mathrm{a}} - \frac{e^{3i\gamma}\log|\mathrm{a}|}{1+e^{3i\gamma}\,\mathrm{a}} \right), \qquad \mathrm{a}^2(\infty) = 2\cos\gamma$$

$$\mathcal{I}_3(\gamma) = \int_{\mathrm{a}^2(-\infty)}^{\mathrm{a}^2(\infty)} d\mathrm{a} \left(\frac{\log\left(1 + e^{-3i\gamma}\,\mathrm{a}\right)}{\mathrm{a}} - \frac{e^{-3i\gamma}\log|\mathrm{a}|}{1+e^{-3i\gamma}\,\mathrm{a}} \right), \qquad \mathrm{a}^2(-\infty) = \sigma = \pm 1$$

- For $0 \leq \gamma \leq \pi$, we prove the general dilogarithm identities

$$\frac{1}{8\pi^2}\mathcal{K}_+(\gamma) = \begin{cases} \frac{1}{24} - \frac{3}{8}\left(\frac{\gamma}{\pi}\right)^2 & 0 \leq \gamma \leq \frac{2\pi}{3} \\ -\frac{23}{24} + \frac{3}{2}\left(\frac{\gamma}{\pi}\right) - \frac{3}{8}\left(\frac{\gamma}{\pi}\right)^2 & \frac{2\pi}{3} \leq \gamma \leq \pi \end{cases}$$

$$\frac{1}{8\pi^2}\mathcal{K}_-(\gamma) = \begin{cases} \frac{1}{6} - \frac{3}{4}\left(\frac{\gamma}{\pi}\right) - \frac{3}{8}\left(\frac{\gamma}{\pi}\right)^2 & 0 \leq \gamma \leq \frac{\pi}{3} \\ -\frac{1}{3} + \frac{3}{4}\left(\frac{\gamma}{\pi}\right) - \frac{3}{8}\left(\frac{\gamma}{\pi}\right)^2 & \frac{\pi}{3} \leq \gamma \leq \pi \end{cases}$$

- Explicitly, the fractional part of $\Delta_{1,s}$ is

$$\mathrm{frac}(\Delta_{1,s}) = \begin{cases} \frac{1}{8\pi^2}\mathcal{K}_-(\frac{2\pi}{3}) \;=\; 0, & s = 1,2 \bmod 3, \qquad \Delta_{1,1} = \Delta_{1,2} = 0 \\ \frac{1}{2} - \frac{1}{8\pi^2}\mathcal{K}_-(0) = \frac{1}{3}, & s = 0 \bmod 3, \qquad \Delta_{1,3} = \frac{1}{3} \end{cases}$$

Solution of TBA Equations for Excitations

- TBA equations can be solved analytically (KlümperP92) for the universal finite size corrections. This yields central charges c, conformal weights $\Delta_{r,s}$ and the conformal spectra of excited states.

- This data allows to build the complete conformal partition functions on the strip and torus. The strip partition functions (with one nontrivial boundary) are precisely the conformal characters

$$\boxed{\chi(q) = q^{-c/24+\Delta} \sum_E d_E\, q^E}$$

- The calculations proceed by using the fact that the arguments of the logarithms are Analytic and Non-Zero once any zeros are removed. After taking the continuum scaling limit in the TBA, dilogarithm techniques yield analytic expressions for the conformal data. The calculation needs the following information:

(a) Analyticity strips.

(b) Braid asymptotic limits $\lim_{u\to\pm i\infty} T(u)$.

(c) Bulk asymptotic limit $\lim_{v\to 0} T(u+iv)$ where $u\in\mathbb{R}$ is in the center of the anayticity strip.

(d) Excitation quantum numbers.

(e) Patterns of zeros in the complex u plane/selection rules.
   – The exact location of the zeros is not needed only their relative ordering.

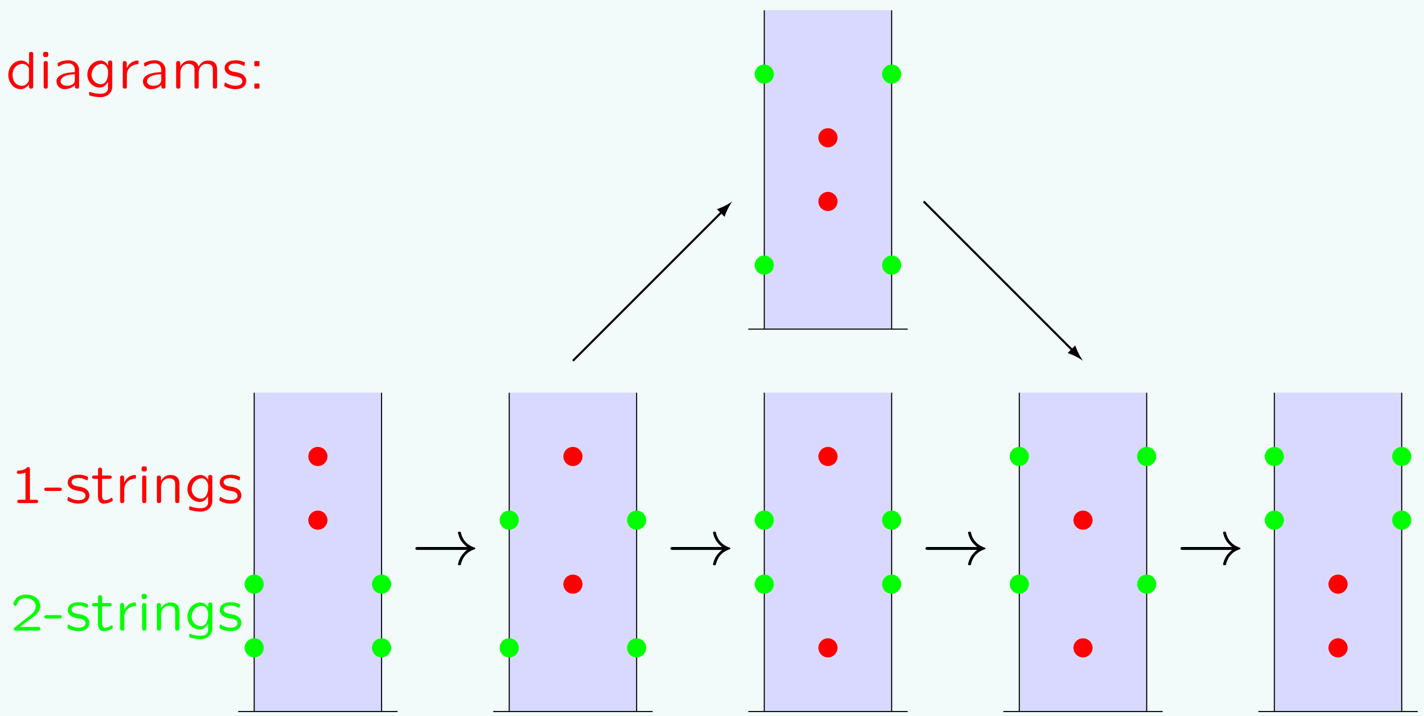
- The most difficult part of the analysis is finding empirical patterns of zeros/selection rules. This combinatorial problem is solved using q-binomials and skew q-binomials.

q-Binomials (Single Column Diagrams)

- The q-binomials are

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \sum_{I_1=0}^{n} \sum_{I_2=0}^{I_1} \cdots \sum_{I_m=0}^{I_{m-1}} q^{I_1+\ldots+I_m} = \begin{cases} \frac{(q)_{m+n}}{(q)_m (q)_n}, & m,n \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad (a)_n = \prod_{k=0}^{n-1} (1-aq^k)$$

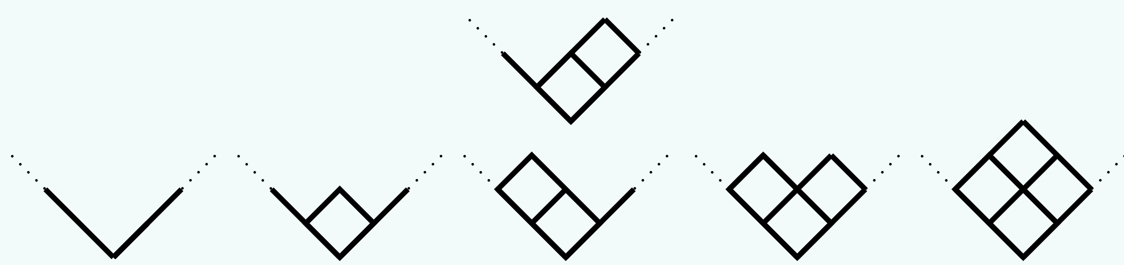
Single column diagrams:



$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

- The q-binomials are realized as sums over partitions or Young/Maya diagrams:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \sum_{\emptyset \subseteq Y \subseteq Y_{2,2}} q^{E(Y)} =$$



$$1 + q + 2q^2 + q^3 + q^4$$

$E(Y) = \#$ boxes

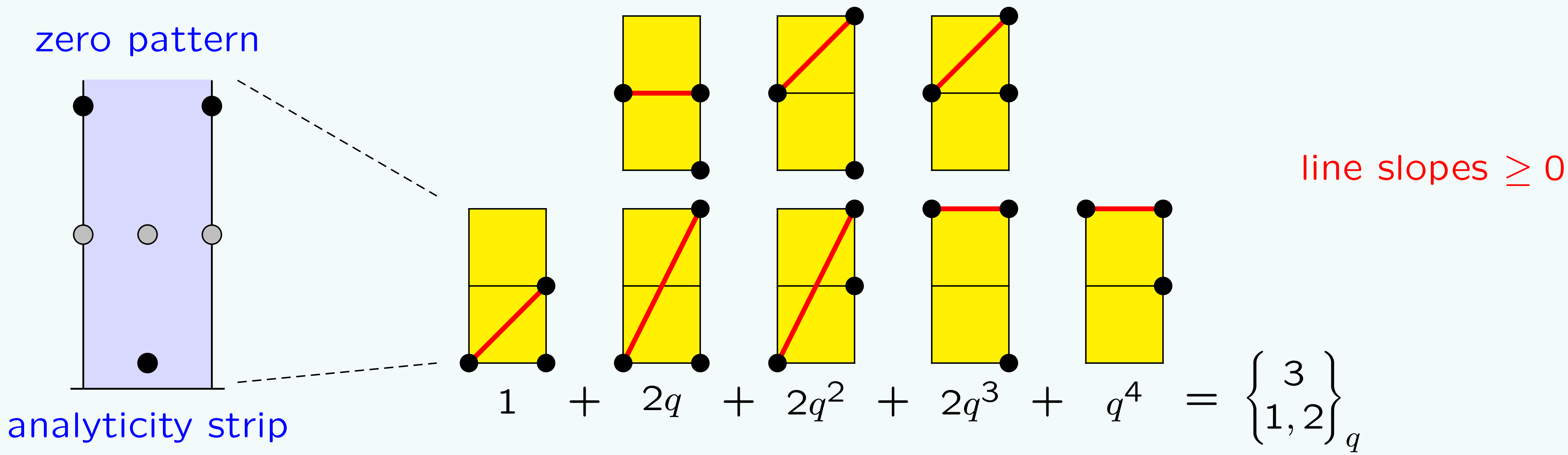
$Y_{m,n} = m \times n$ Young diagrams

Skew q-Binomials (Double Row Diagrams)

- The skew q-binomials are (PRasm2007)

$$\boxed{\left\{ \begin{matrix} M \\ m, n \end{matrix} \right\}_q = q^{-M+n} \left(\begin{bmatrix} M \\ m \end{bmatrix}_q \begin{bmatrix} M+1 \\ n+1 \end{bmatrix}_q - \begin{bmatrix} M+1 \\ m \end{bmatrix}_q \begin{bmatrix} M \\ n+1 \end{bmatrix}_q \right), \qquad 0 \le m \le n \le M}$$

- The skew q-binomials are enumerated by double column diagrams with dominance



Skew q-Binomials and Skew Partitions

- A partition λ is equivalent to a Young diagram Y. A skew Young diagram Y_2/Y_1 is equivalent to the pair (Y_1, Y_2) with $Y_1 \subseteq Y_2$. Let us define

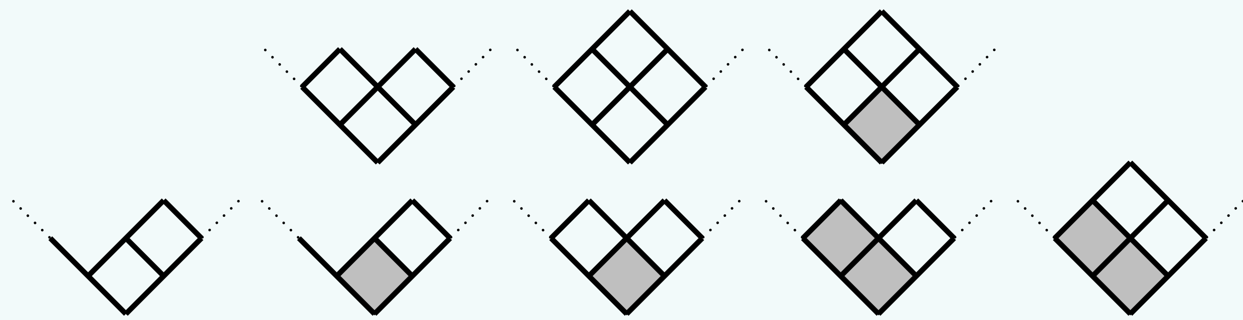
$$\boxed{\begin{gathered} E(Y) = \text{Energy} = \{\#\text{ of boxes in the Young digram } Y\} \\ Y_{m,n} = \{m \times n \text{ rectangular Young diagram}\} \end{gathered}}$$

- A skew q-binomial can be written as an energy weighted sum over skew Young diagrams

$$\boxed{\left\{ {M \atop m,n} \right\}_q = q^{(m-n)n} \sum_{\substack{Y_1 \subseteq Y_2 \\ \emptyset \subseteq Y_1 \subseteq Y_{M-m,m} \\ Y_{n-m,n} \subseteq Y_2 \subseteq Y_{M-m,n}}} q^{E(Y_1)+E(Y_2)}, \qquad 0 \le m \le n \le M}$$

- The bijection is implemented by interpreting the left and right column (particle) configurations in the double column diagrams as Maya diagrams and using the standard bijection between Maya diagrams and Young diagrams. For example, shading Y_1, gives

$$\left\{ {3 \atop 1,2} \right\}_q = q^{-2} \Bigg[$$

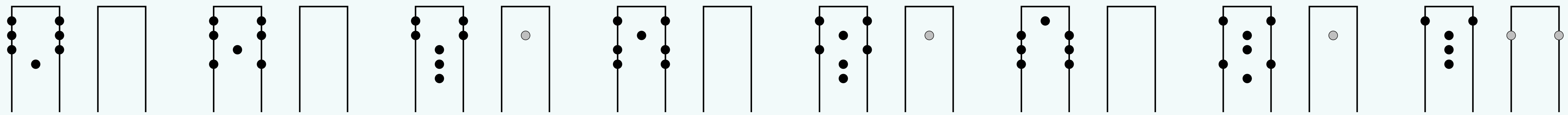


$$1 \quad + \quad 2q \quad + \quad 2q^2 \quad + \quad 2q^3 \quad + \quad q^4 \Bigg], \qquad \begin{gathered} \emptyset \subseteq Y_1 \subseteq Y_{2,1} \\ Y_{1,2} \subseteq Y_2 \subseteq Y_{2,2} \end{gathered}$$

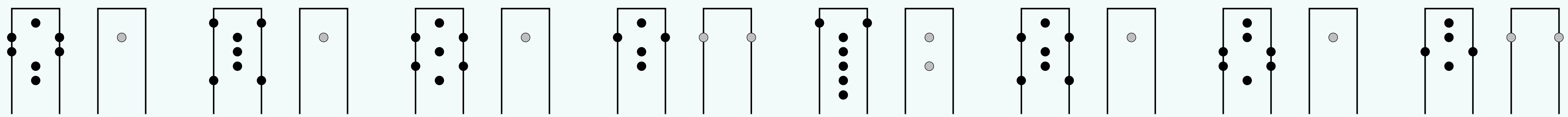
Part 4

Strip Conformal Partition Functions
= Finitized Characters

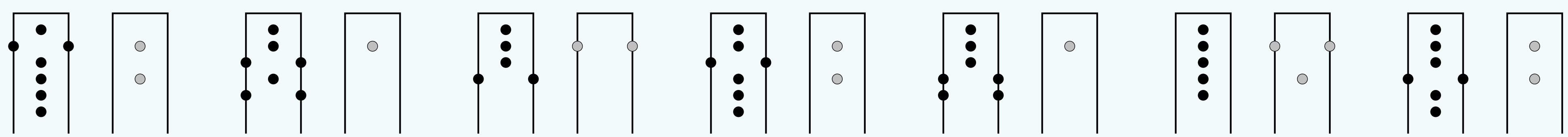
Selection Rules: 28 Patterns of Zeros for $\mathrm{V}^{d=2}_{N=8}$ $(\Delta_{1,3}=\frac{1}{3})$



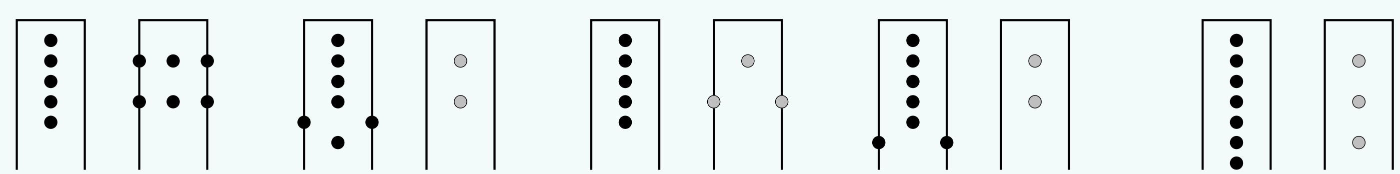
k^j :	1	2	1, 2, 3	3	1, 2, 4	4	1, 3, 4	1, 2, 3
ℓ^i :	$*$	$*$	1, 1	$*$	1, 1	$*$	1, 1	$*$
q^{Δ} :	$q^{1/3}$	$q^{4/3}$	$q^{7/3}$	$q^{7/3}$	$q^{10/3}$	$q^{10/3}$	$q^{13/3}$	$q^{13/3}$



k^j :	1, 2, 5	2, 3, 4	1, 3, 5	1, 2, 4	1, 2, 3, 4, 5	2, 3, 5	1, 4, 5	1, 3, 4
ℓ^i :	1, 1	1, 1	1, 1	$*$	1, 1, 2, 2	1, 1	1, 1	$*$
q^{Δ} :	$q^{13/3}$	$q^{16/3}$	$q^{16/3}$	$q^{16/3}$	$q^{19/3}$	$q^{19/3}$	$q^{19/3}$	$q^{19/3}$



k^j :	1, 2, 3, 4, 6	2, 4, 5	2, 3, 4	1, 2, 3, 5, 6	3, 4, 5	1, 2, 3, 4, 5	1, 2, 4, 5, 6
ℓ^i :	1, 1, 2, 2	1, 1	$*$	1, 1, 2, 2	1, 1	1, 1	1, 1, 2, 2
q^{Δ} :	$q^{22/3}$	$q^{22/3}$	$q^{22/3}$	$q^{25/3}$	$q^{25/3}$	$q^{25/3}$	$q^{28/3}$



$$= \bigcup_{i=0}^{\frac{N-d}{2}} \bigcup_{j=0}^{\left\lfloor \frac{1}{2}\left(\frac{N-d}{2}-i\right)\right\rfloor} \mathcal{S}_{2(i+j+t)+1}^{\frac{N+t}{2}+i} \, \mathcal{D}_{i,i+t}^{i+j+t}$$

$t=(d-2)/3$ $\boldsymbol{K}$ $\boldsymbol{d}^1$

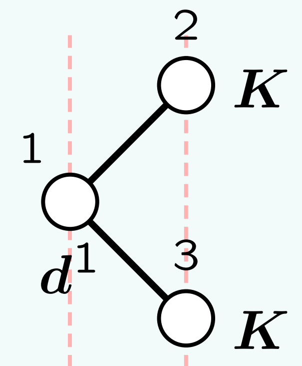
k^j :	1, 2, 3, 4, 5	1, 3, 4, 5, 6	1, 2, 3, 4, 5	2, 3, 4, 5, 6	1, 2, 3, 4, 5, 6, 7
ℓ^i :	1, 2	1, 1, 2, 2	2, 2	1, 1, 2, 2	1, 1, 2, 2, 3, 3
q^{Δ} :	$q^{28/3}$	$q^{31/3}$	$q^{31/3}$	$q^{34/3}$	$q^{37/3}$

Percolation: Finitized Characters on the Strip

- For bond percolation $\mathcal{LM}(2,3)$ on the strip the D_3 Y-system is

$$\boxed{\boldsymbol{d}_0^1\boldsymbol{d}_1^1 = (\boldsymbol{I}+\boldsymbol{K}_0)^2, \qquad \boldsymbol{K}_0\boldsymbol{K}_1 = \boldsymbol{I}+\boldsymbol{d}_1^1}$$

D_3 :



- For this model, there are 2 analyticity strips with 1-strings and 2-strings.

For $s = d+1 = 3t+3 = 3,6,9,\ldots$, the combinatorial classification of zero patterns and TBA analysis leads to refined finitized characters (M-DKP2017)

$$\begin{aligned}
Z^{(N)}_{(1,1)|(1,s)}(q) &= \chi^{(N)}_{1,s}(q) = q^{\frac{(s-1)(s-2)}{6}}\left(\begin{bmatrix} N \\ \frac{N+1-s}{2}\end{bmatrix}_q - q^s \begin{bmatrix} N \\ \frac{N-1-s}{2}\end{bmatrix}_q\right) \qquad \text{bosonic (PRZ2006)} \\
&= q^{\frac{(s-1)(s-2)}{6}} \sum_{i,j} q^{i^2+2ij+2j^2+i+2j+t(2i+3j)} \begin{Bmatrix} i+j+t \\ i,\ i+t \end{Bmatrix}_q \begin{bmatrix} \frac{N+t}{2}+i \\ 2(i+j+t)+1 \end{bmatrix}_q \qquad \text{fermionic} \\
&\to \chi_{1,s}(q), \qquad N\to\infty
\end{aligned}$$

Similar but more complicated expressions are obtained for $s = 1,2 \bmod 3$.

- In general, the branching node of the $D_{p'}$ Dynkin diagram is associated with skew q-binomials. All other nodes are associated with q-binomials.

- Skew q-binomials are differences of products of two q-binomials. So the summand breaks up into products of three q-binomials. We need an identity to do one sum leaving a sum over products of two q-binomials. This last sum is performed using the q-Vandermonde identity leaving a single q-binomial.

Elementary Proof of Binomial Identity

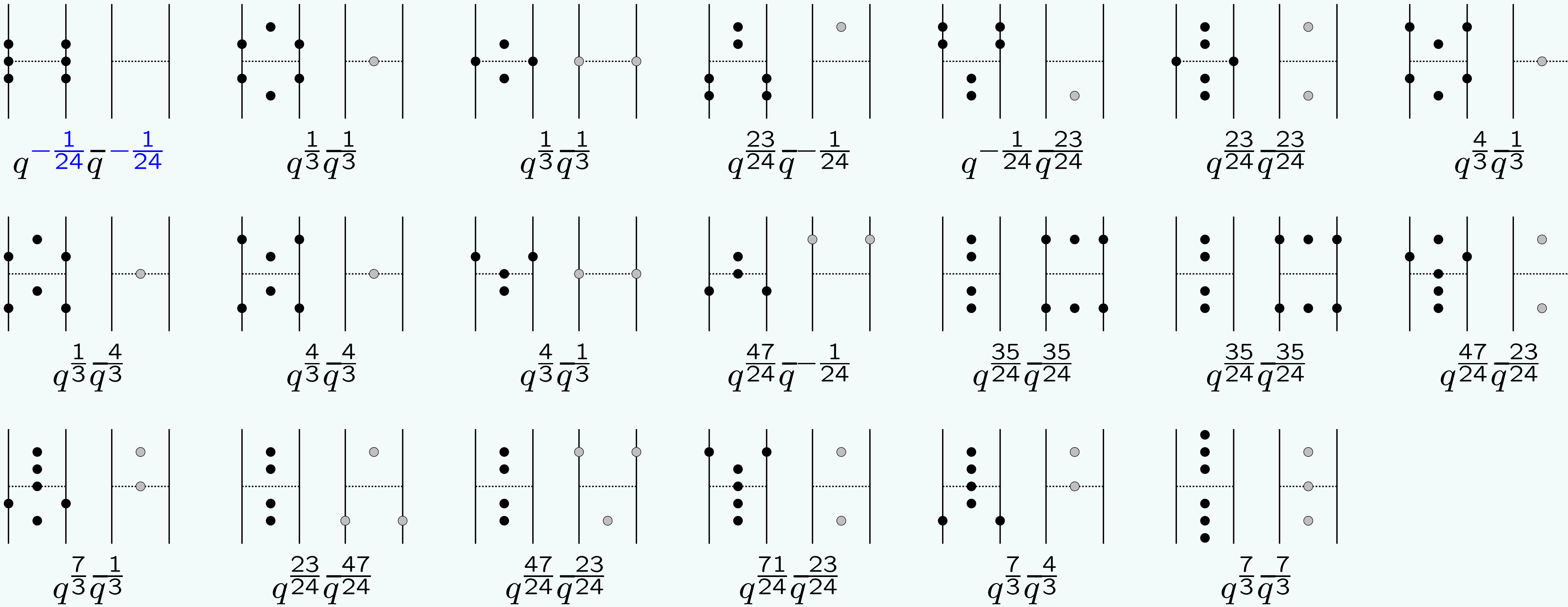
- The key identity is the q-Saalschütz identity (Jackson1910). Our elementary proof similar to (Gould1972) is as follows:

$$
\begin{aligned}
\begin{bmatrix} m \\ p \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q &= \begin{bmatrix} m-n+r+n-r \\ p \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q = \sum_k \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} n-r \\ k \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q q^{k(m-n+r-p+k)} \\
&= \sum_k \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} r+k \\ r \end{bmatrix}_q \begin{bmatrix} n \\ r+k \end{bmatrix}_q q^{k(m-n+r-p+k)} \\
&= \sum_k \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} r+k \\ r \end{bmatrix}_q q^{k(m-n+r-p+k)} \sum_j (-1)^j \begin{bmatrix} n+p-k-j \\ p+r \end{bmatrix}_q \begin{bmatrix} p-k \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j+1)+j(r+k)} \\
&= \sum_k \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} r+k \\ r \end{bmatrix}_q q^{k(m-n+r-p+k)} \sum_i (-1)^{p-k-i} \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} p-k \\ i \end{bmatrix}_q q^{\frac{1}{2}(p-k-i)(p+k-i+1+2r)} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \sum_k (-1)^{p-k-i} \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} p-k \\ i \end{bmatrix}_q q^{k(m-n+r-p+k)+\frac{1}{2}(p-k-i)(p+k-i+1+2r)} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} m-n+r \\ i \end{bmatrix}_q \sum_k (-1)^{p-k-i} \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} m-n+r-i \\ p-i-k \end{bmatrix}_q q^{k(m-n+r-p+k)+\frac{1}{2}(p-k-i)(p+k-i+1+2r)} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} m-n+r \\ i \end{bmatrix}_q (-1)^{p-i} \begin{bmatrix} m-n-i-1 \\ p-i \end{bmatrix}_q q^{\frac{1}{2}i(i-1)+\frac{1}{2}p(p+1)+pr-pi-ri} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} m-n+r \\ i \end{bmatrix}_q \begin{bmatrix} n-m+p \\ p-i \end{bmatrix}_q q^{\frac{1}{2}i(i-1)+\frac{1}{2}p(p+1)+pr-pi-ri-\frac{1}{2}(p-i)(p-i-1)+(p-i)(m-n-i-1)} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} m-n+r \\ i \end{bmatrix}_q \begin{bmatrix} n-m+p \\ n-m+i \end{bmatrix}_q q^{i^2+i(n-m-p-r)+mp-np+pr}
\end{aligned}
$$

Part 5

Modular Invariant Partition Function on the Torus

20 Zero Patterns for $\mathrm{W}^{d=0}_{N=6}$ on the Cylinder $(\Delta_{0,0}=\bar{\Delta}_{0,0}=-\frac{1}{24})$



- **Finitized torus partition function** $Z^N(q)$: $\quad q=\exp\left(-2\pi i\,\frac{M}{N}\,e^{-3iu}\right)$

$$Z^{N=6}_{d=0}(q) \;=\; \sum_{\text{states in } \mathrm{W}^0_6} q^{\Delta}\bar{q}^{\bar{\Delta}} \;=\; \begin{cases} q^{-\frac{1}{24}}\bar{q}^{-\frac{1}{24}}+q^{\frac{1}{3}}\bar{q}^{\frac{1}{3}}+q^{\frac{1}{3}}\bar{q}^{\frac{1}{3}}+q^{\frac{23}{24}}\bar{q}^{-\frac{1}{24}} \\ \qquad +q^{-\frac{1}{24}}\bar{q}^{\frac{23}{24}}+q^{\frac{23}{24}}\bar{q}^{\frac{23}{24}}+q^{\frac{4}{3}}\bar{q}^{\frac{1}{3}}+\ldots \end{cases}$$

$$Z^N(q)=\sum_{d=0,2,4,\ldots,N}(2-\delta_{d,0})Z^N_d(q), \qquad Z(q)=\lim_{N\to\infty}Z^N(q)$$

Modular Invariant Torus Partition Function

- A similar TBA analysis generalizes to the periodic single row transfer matrix on the cylinder.
- For M, N even the resulting torus partition function is

$$\boxed{Z(q) = |\varkappa_0(q)|^2 + 2\varkappa_1(q)\varkappa_5(\bar{q}) + 2|\varkappa_2(q)|^2 + 2|\varkappa_3(q)|^2 + 2|\varkappa_4(q)|^2 + 2\varkappa_5(q)\varkappa_1(\bar{q}) + |\varkappa_6(q)|^2}$$

This is a nondiagonal (D-type) modular invariant in specialized affine $u(1)$ characters

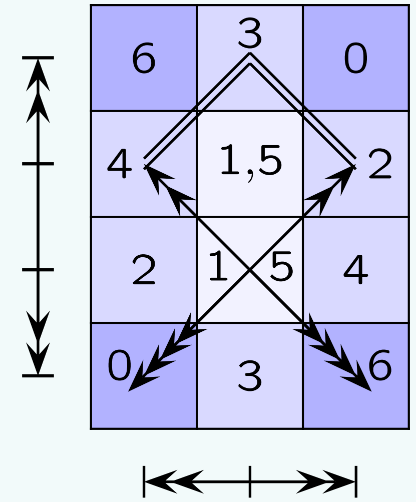
$$\boxed{\varkappa_j(q) = \varkappa_j^n(q,1), \qquad \varkappa_j^n(q,z) = \frac{\Theta_{j,n}(q,z)}{q^{1/24}(q)_\infty} = \frac{q^{-1/24}}{(q)_\infty}\sum_{k\in\mathbb{Z}} z^k q^{(j+2kn)^2/4n}, \qquad n = pp' = 6}$$

The modular nome is

$$\boxed{q = \exp\left(-2\pi i\delta\, e^{-3iu}\right), \qquad \lim_{M,N\to\infty}\frac{M}{N} = \delta = \text{aspect ratio}}$$

- The modular invariant is naturally encoded by the coset graph $A_2^{(2)} \otimes A_3^{(2)}/\mathbb{Z}_2$ in A-type twisted affine Dynkin diagrams.
- A simple identity gives the trivial Virasoro character

$$\varkappa_1(q) - \varkappa_5(q) = 1 = \mathrm{ch}^{2,3}_{1,1}(q)$$



s			
3	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$
2	$\frac{5}{8}$	0	$\frac{1}{8}$
1	$\frac{1}{8}$	0	$\frac{5}{8}$
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$
	0	1	2 r

$u(1)$ Kac Table

Summary

- Critical bond percolation on the square lattice is the $\mathcal{LM}(2,3)$ logarithmic minimal model.
- The universal conformal spectra is obtained analytically. Solving the Y-system on the strip gives $c=0$, $\Delta_{1,s}=[(3-2s)^2-1]/24$, $s\in\mathbb{N}$ and finitized conformal characters in agreement with Pearce-Rasmussen-Zuber (2006).
- Solving the Y-system on the torus yields a D-type $u(1)$ MIPF encoded by the twisted affine coset graph $A_2^{(2)}\otimes A_3^{(2)}/\mathbb{Z}_2$.

Some open problems:

- It remains to extend this to the known general (r,s)-type Kac boundary conditions and Robin boundary conditions for half-integer Kac labels (r,s).
- The natural conjecture for the (p,p') torus partition function is the $u(1)$ MIPF

$$\boxed{Z_{p,p'}(q)=Z^{\mathrm{Circ}}_{p,p'}(q)=\sum_{j=0}^{2n-1}\varkappa_j^n(q)\varkappa^n_{\omega_0 j}(\bar q),\qquad n=pp'}$$

where $\omega_0 j$ is the Bezout conjugate. This is the MIPF of a boson on the circle S^1, with radius of compactification $R=\sqrt{2p'/p}$ and effective central charge $c_{\mathrm{eff}}=1$. But the combinatorial description/analysis of the D-type TBA for general (p,p') looks formidable!

Warnaar's Elegant Proof of Binomial Identity

- The q-Saalschütz identity (Jackson1910) can be written as

$$\boxed{\sum_{i=0}^{M} q^{i(i+\ell)} \begin{bmatrix} L_1+L_2+M-i \\ M-i \end{bmatrix}_q \begin{bmatrix} L_1 \\ i+\ell \end{bmatrix}_q \begin{bmatrix} L_2 \\ i \end{bmatrix}_q = \begin{bmatrix} L_1+M \\ M+\ell \end{bmatrix}_q \begin{bmatrix} L_2+M+\ell \\ M \end{bmatrix}_q}$$

with $m = L_1 + M$, $p = L_1 - \ell$, $n = L_2 + M + \ell$, $r = L_2 + \ell$.

- A more general identity (Carlitz1974) uses q-hypergeometric functions

$$\boxed{S_n(a,b,c;q) = {}_3\phi_2\left[\begin{matrix} a, b, q^{-n} \\ c, abq^{1-n}/c \end{matrix}; q, q\right] := \sum_{k=0}^{n} \frac{(a)_k (b)_k (q^{-n})_k\, q^k}{(q)_k (c)_k (abq^{1-n}/c)_k} = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n}}$$

with $a, b, c \in \mathbb{C}$ and the shifted q-factorials

$$(a)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \qquad (a)_n = \frac{(a)_\infty}{(aq^n)_\infty}$$

- Denoting the summand by $S_{n,k}(a,b,c;q)$ we have $S_{n,k}(a,b,c;q) = 0$ for $k > n$ and the recursion

$$\boxed{S_{n,k}(a,b,c;q) = S_{n-1,k}(a,b,c;q) - \frac{(1-a)(1-b)(1-abq/c)q^{1-n}}{(1-c)(1-abq^{1-n}/c)(1-abq^{2-n}/c)} S_{n-1,k-1}(aq,bq,cq;q)}$$

Together with the initial conditions $S_{n,0}(a,b,c;q) = 1$ this uniquely determines $S_{n,k}(a,b,c;q)$.

- Summing over k gives

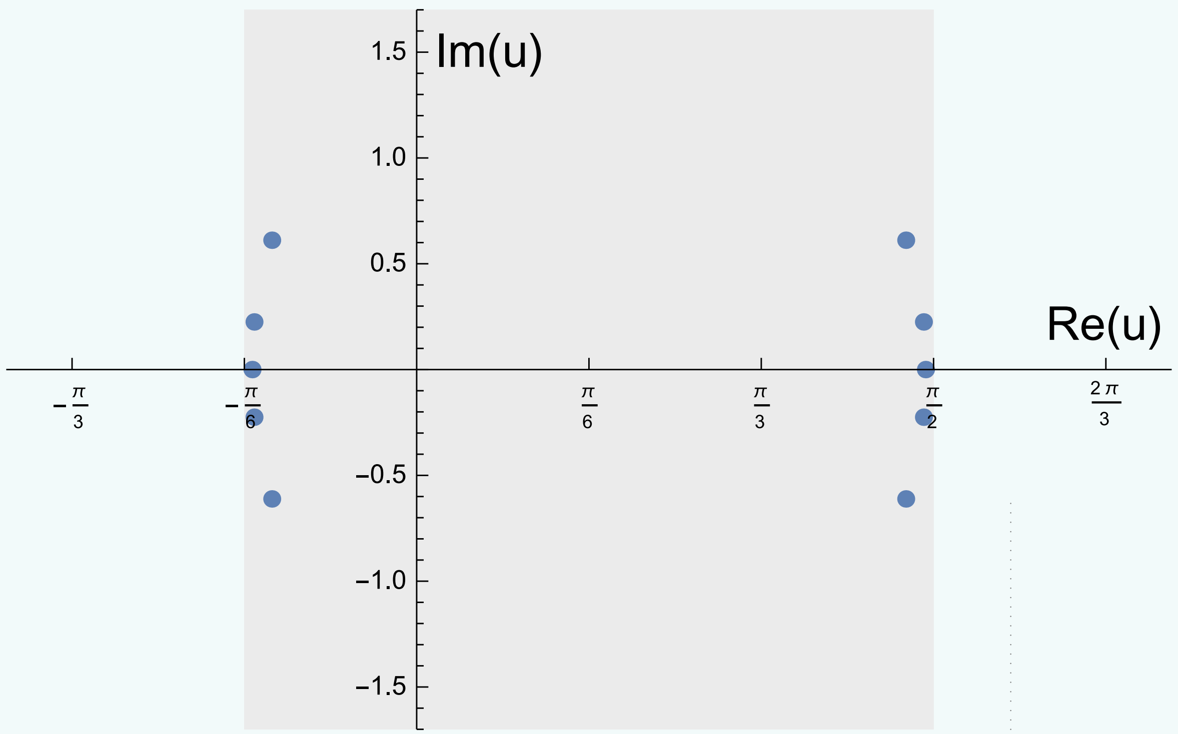
$$\boxed{S_n(a,b,c;q) = S_{n-1}(a,b,c;q) - \frac{(1-a)(1-b)(1-abq/c)q^{1-n}}{(1-c)(1-abq^{1-n}/c)(1-abq^{2-n}/c)} S_{n-1}(aq,bq,cq;q)}$$

It is trivial to check that, with the initial condition $S_0(a,b,c;q) = 1$, this is satisfied by the right-hand side of the identity.

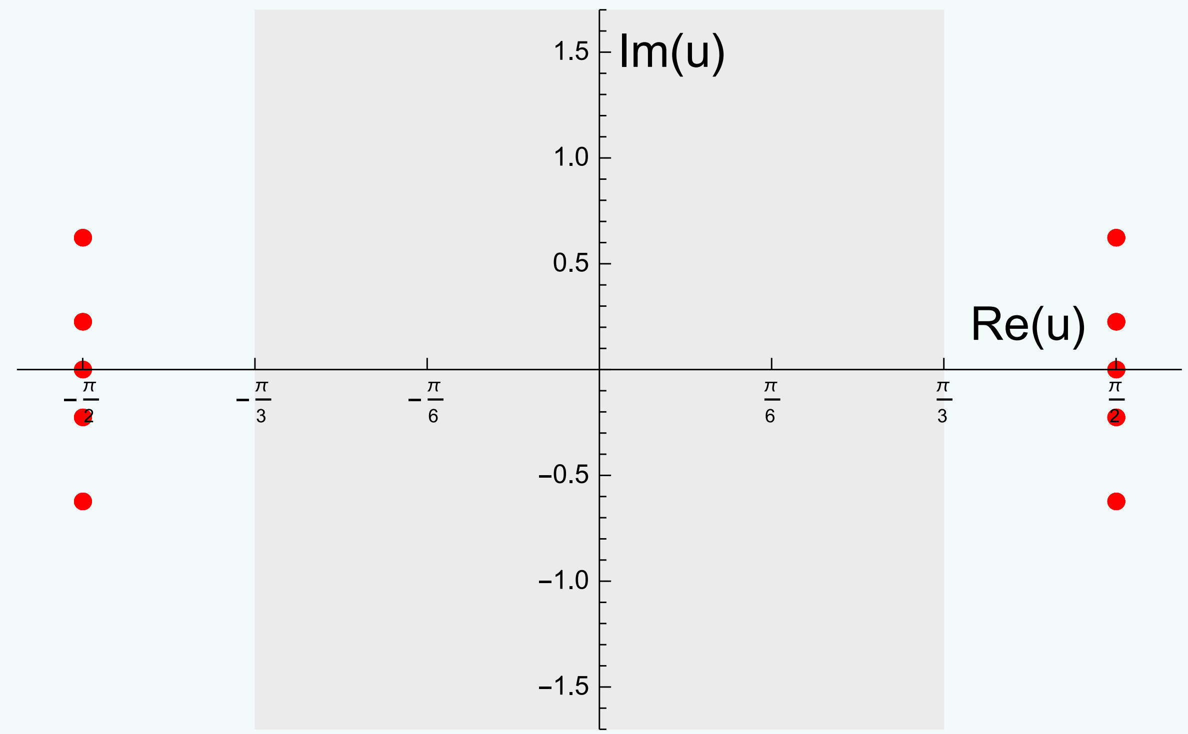
Ground State on the Cylinder for $d = 0$

- **Patterns of zeros** and **analyticity strips** for $d = 0$, $N = 10$ and $\alpha = 2$:

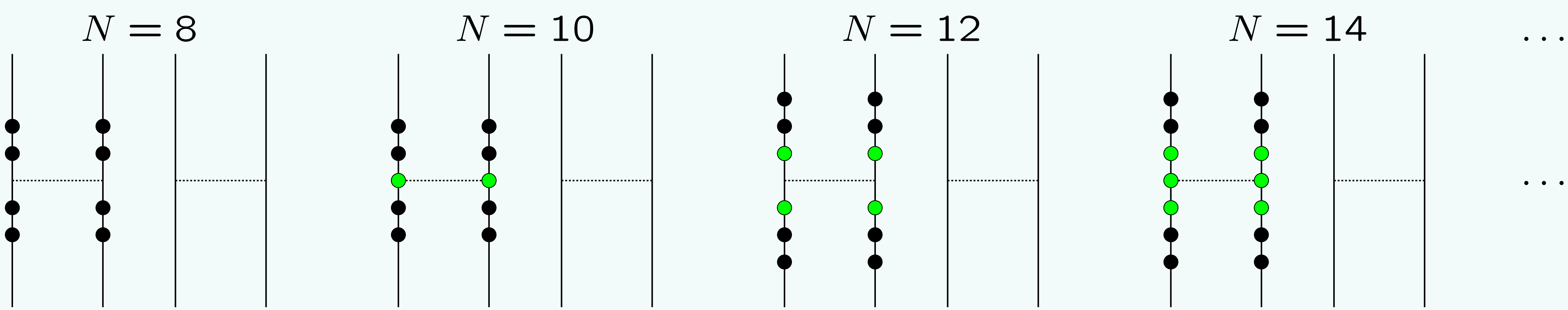
$T^1(u)$:



$T^2(u)$:



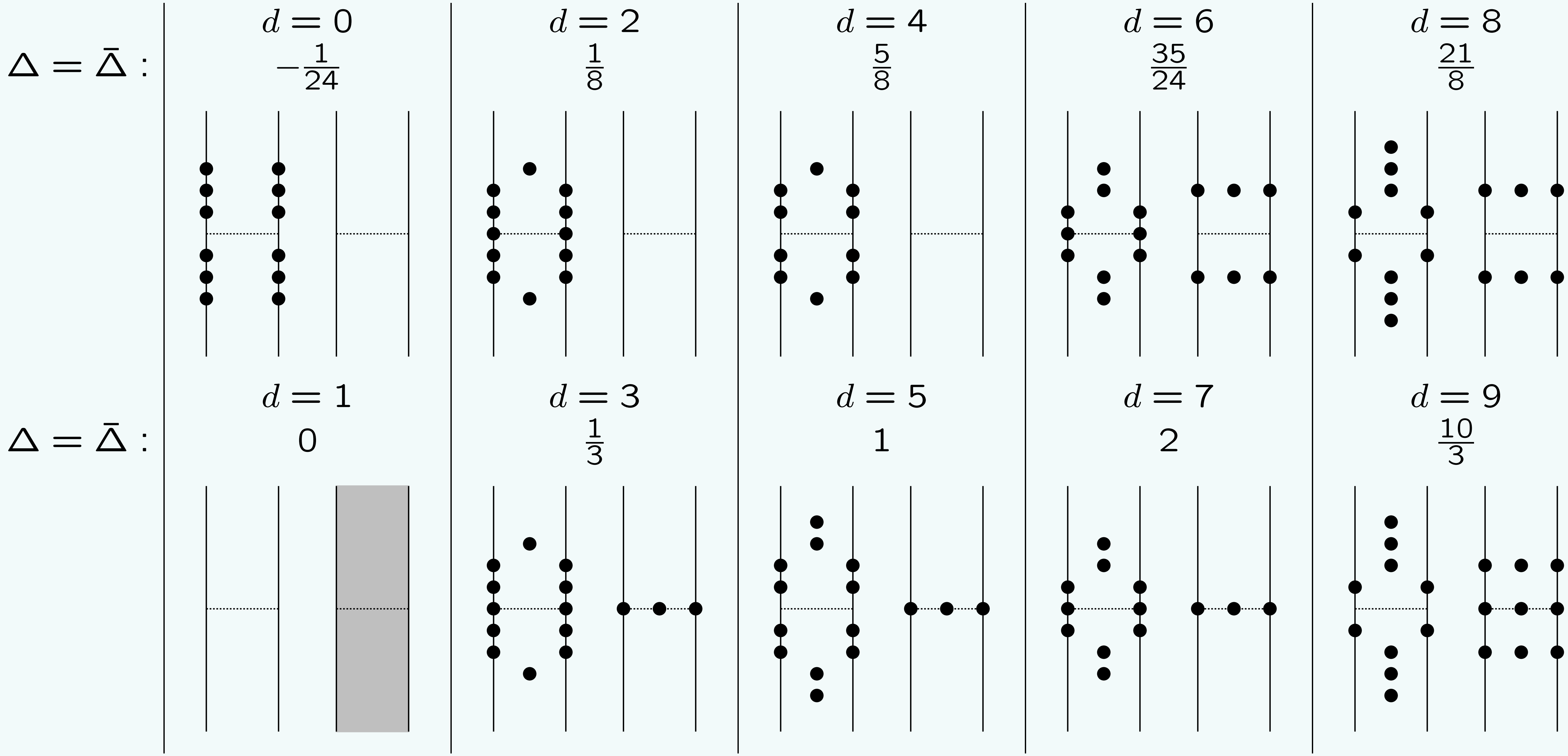
- Sequences of ground states patterns of zeros:



2-strings = ● ● = filling of the Fermi sea

Ground State Zero Patterns on Cylinder

- Patterns of zeros and conformal weights for $N = 12$, 13 repectively



- The ground state in the $d = 1$ sector on the cylinder and in the $d = 0$ sector on the strip is the Razumov-Stroganov eigenvalue. For this trivial eigenvalue $c = \Delta = \bar{\Delta} = 0$ and there are no finite-size corrections

$$\boxed{\begin{array}{lcl} \mathfrak{a}^1(x) = 0 & & T^1(u) = -i^N \left(\dfrac{\sin(u+\frac{\pi}{3})}{\sin\frac{\pi}{3}}\right)^N \\ & \longrightarrow & \\ \mathfrak{a}^2(x) = -1 & & T^2(u) = 0 \end{array}}$$