The Algebraic Construction of Integrable Hierarchies, Solitons and Backlund Transformation

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Discuss the General structure of time evolution integrable equations associated to graded Affine Lie algebraic structure, e.g., sinh-Gordon, mKdV, etc.

Representation Theory of Infinite Dimensional Algebras and the Systematic Construction of Soliton Solutions.

Systematic Construction of Backlund Transformation for the entire Hierarchy and Integrable Defects
Start with affine Lie Algebra $\hat{G}$

Decompose $\hat{G}$ into graded subspaces, e.g., $\hat{G} = \bigoplus_i G_i$, such that $[G_i, G_j] \subset G_{i+j}$.

Define constant grade 1 operator $E = E^{(1)} \in G_1$ and define $\mathcal{K} = Kernel = \{x \in \hat{G} / [x, E] = 0\}$

Decompose $G_0 = \mathcal{K} \oplus M$
Define Lax operator

\[ L = \partial_x + E^{(1)} + A_0, \quad A_0 \in \mathcal{M} \subset G_0 \quad \text{Image} \]

2 - Dim. Gauge potentials

\[ A_x = E + A_0, \quad A_{t_{MN}} = D^{(N)} + D^{(N-1)} + \ldots + D^{(0)} + \ldots D^{(-M)} \]

Zero Curvature Equation for Hierarchy

\[ [\partial_x + E^{(1)} + A_0, \partial_{t_{MN}} + D^{(N)} + D^{(N-1)} + \ldots + D^{(0)} + \ldots D^{(-M)}] = 0 \]

\[ D^{(j)} \in G_j. \]
Decompose and solve grade by grade, i.e.,

\[
\begin{align*}
[E^{(1)}, D^{(N)}] &= 0, \\
[E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0, \\
&\vdots \\
[E^{(1)}, D^{(-1)}] + [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_{MN}} A_0 &= 0, \\
&\vdots \\
[A_0, D^{(-M)}] + \partial_x D^{(-M)} &= 0,
\end{align*}
\]

Solving recursively for \(D^{(i)}\) we get the eqn. of motion

\[
\partial_{t_{MN}} A_0 - \partial_x D^{(0)} - [A_0, D^{(0)}] - [E^{(1)}, D^{(-1)}] = 0,
\]
Example: The mKdV Hierarchy

- Choose $G = \mathfrak{sl}(2)$ with generators $\{h, E_{\pm \alpha}\}$

- Grading operator $Q$, e.g. $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h$

- *semi-simple* element $E = E^{(1)} = E_{\alpha} + \lambda E_{-\alpha}$

- Decomposition of Affine Lie Algebra into graded subspaces,

  \[ G_{2m} = \{ h^{(m)} = \lambda^m h \}, \]
  \[ G_{2m+1} = \{ \lambda^m (E_{\alpha} + \lambda E_{-\alpha}), \lambda^m (E_{\alpha} - \lambda E_{-\alpha}) \} \]

  \( m = 0, \pm 1, \pm 2, \ldots \) where $[G_i, G_j] \subset G_{i+j}$. 
For $M = 0$, Zero Curvature can be decomposed and solved grade by grade, i.e.,

\[
\begin{align*}
[E^{(1)}, D^{(N)}] &= 0, \\
[E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0, \\
&\quad \vdots \\
[A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_t N A_0 &= 0,
\end{align*}
\]

In particular, highest grade component, i.e.,

\[
[E^{(1)}, D^{(N)}] = 0,
\]

implies $D^{(N)} = E^{(N)} \in \mathcal{K}_{2n+1}$ is const. and therefore $N = 2n + 1$.

Solving recursively for $D^{(i)}$ we get the eqn. of motion

\[
\partial_t N A_0 - \partial_x D^{(0)} - [A_0, D^{(0)}] = 0,
\]
Positive mKdV Hierarchy

Examples: $A_0 = v(x, t_N) h$,

$N = 3,$

$$4 \partial_{t_3} v = v_{3x} - 6v^2 v_x, \quad mKdV$$

$N = 5,$

$$16 \partial_{t_5} v = v_{5x} - 10v^2 v_{3x} - 40v v_x v_{2x} - 10v_x^3 + 30v^4 v_x,$$

$N = 7,$

$$64 \partial_{t_7} v = v_{7x} - 182v_x v_{2x}^2 - 126 v_x^2 v_{3x} - 140v v_{2x} v_{3x}$$

$$- 84v v_x v_{4x} - 14v^2 v_{5x} + 420 v^2 v_{3x} + 560 v^3 v_x v_{2x}$$

$$+ 70v^4 v_{3x} - 140v^6 v_x$$

$\cdots$ etc
Remark

- Vacuum Solution is $\nu = \text{const} = 0$
- Zero curvature for vacuum solution becomes
  \[
  [\partial_x + E^{(1)}, \partial_{tN} + E^{(N)}] = 0, \quad [E^{(1)}, E^{(N)}] = 0
  \]
  and imply pure gauge potentials, i.e.,
  \[
  A_{x,\text{vac}} = E^{(1)} = T_0^{-1} \partial_x T_0, \quad A_{tN,\text{vac}} = E^{(N)} = T_0^{-1} \partial_{tN} T_0
  \]
- Where
  \[
  T_0 = e^{x E^{(1)}} e^{tN E^{(N)}}
  \]
Zero Curvature Equation for \textit{Negative Hierarchy}

\[ \partial_x + E^{(1)} + A_0, \partial_{t_n} + D(-n) + D(-n+1) + \ldots + D(-1) = 0. \]

Lowest grade projection,

\[ \partial_x D(-n) + [A_0, D(-n)] = 0 \]

yields a nonlocal equation for \( D(-n) \). \textbf{No condition upon} \( n \).
The second lowest projection of grade $-n + 1$ leads to
\[ \partial_x D^{(-n+1)} + [A_0, D^{(-n+1)}] + [E^{(1)}, D^{(-n)}] = 0 \]
and determines $D^{(-n+1)}$.

The same mechanism works recursively until we reach the zero grade equation
\[ \partial_{t_{-n}} A_0 + [E^{(1)}, D^{(-1)}] = 0 \]
which gives the *time evolution* for the field in $A_0$ according to time $t_{-n}$. 
Simplest Example $t_{-n} = t_{-1}$.

\[ \partial_x D^{(-1)} + [A_0, D^{(-1)}] = 0, \]

(1)

\[ \partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] = 0. \]

Compact Solution is

\[ D^{(-1)} = B^{-1} E^{(-1)} B, \quad A_0 = B^{-1} \partial_x B, \quad B = \exp(G_0) \]

The time evolution is then given by the Leznov-Saveliev equation,

\[ \partial_{t_{-1}} \left( B^{-1} \partial_x B \right) = [E^{(1)}, B^{-1} E^{(-1)} B] \]
For $\hat{sl}(2)$ with principal gradation $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h$, yields the sinh-Gordon equation (relativistic)

$$\partial_{t_{-1}} \partial_x \phi = e^{2\phi} - e^{-2\phi}, \quad B = e^{\phi h}.$$

where $t_{-1} = z$, $x = \bar{z}$, $A_0 = vh = \partial_x \phi h$.

No restriction for Negative even Hierarchy
Next simplest example $t = t_{-2}$

\[
\partial_x D^{(-2)} + [A_0, D^{(-2)}] = 0,
\]
\[
\partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] = 0,
\]
\[
\partial_{t_{-2}} A_0 - [E^{(1)}, D^{(-1)}] = 0.
\]

Propose solution of the form

\[
D^{(-2)} = c_{-2} \lambda^{-1} h,
\]
\[
D^{(-1)} = a_{-1} \left( \lambda^{-1} E_{\alpha} + E_{-\alpha} \right) + b_{-1} \left( \lambda^{-1} E_{\alpha} - E_{-\alpha} \right).
\]
Get $c_{-2} = \text{const}$ and

\[
\begin{align*}
    a_{-1} + b_{-1} &= 2c_{-2} \exp(-2d^{-1}v)d^{-1} \left(\exp(2d^{-1}v)\right), \\
    a_{-1} - b_{-1} &= -2c_{-2} \exp(2d^{-1}v)d^{-1} \left(\exp(-2d^{-1}v)\right),
\end{align*}
\]

where $A_0 = vh = \partial_x \phi h$ and $d^{-1}v = \int^x v(x')dx' = \phi$. 

The Algebraic Construction of Integrable Hierarchies, Solitons and Backlund Transformations
Equation of motion is (integral eqn.)

$$\partial_{t_{−2}} \nu = -2c_{−2} e^{-2d^{-1} \nu} d^{-1} \left( e^{2d^{-1} \nu} \right) - 2c_{−2} e^{2d^{-1} \nu} d^{-1} \left( e^{-2d^{-1} \nu} \right)$$

where $d^{-1} \nu = \int^x \nu(x') dx' = \phi$. 
Constant Vacuum for $t = t_{-2}$ equation

1) Let $v = 0$, $d^{-1}0 = \alpha = \text{const}$

$$0 + 2c_{-2}e^{-2\alpha} \int e^{2\alpha} + 2c_{-2}e^{2\alpha} \int e^{-2\alpha} \neq 0,$$

for $c_{-2} \neq 0$.

2) $v = v_0$, $d^{-1}v_0 = v_0 x$

$$0 + 2c_{-2}e^{-2v_0 x} \int e^{2v_0 x} + 2c_{-2}e^{2v_0 x} \int e^{-2v_0 x} = 0$$

Notice that, for $c_{-2} \neq 0$, $v = 0$ is not solution of the evolution equation and therefore $A_0 = 0$ does not satisfy the zero curvature representation for $t = t_{-2}$. 
The **Soliton solutions** are constructed from the vacuum solution by **gauge transformation** (which preserves the zero curvature condition), i.e.,

\[ A_\mu = \Theta^{-1} A_{\mu, vac} \Theta + \Theta^{-1} \partial_\mu \Theta, \]

where

\[ A_\mu = T^{-1} \partial_\mu T, \quad T = T_0 \Theta, \quad A_{\mu, vac} = T_0^{-1} \partial_\mu T_0 \]

we may choose \( \Theta = \Theta_+ = e^{\theta_0} e^{\theta_1} \cdots \) or \( \Theta = \Theta_- = e^{\theta_{-1}} e^{\theta_{-2}} \cdots \), \( \theta_i \in G_i \).
It then follows that \( T = \Theta_+ T_0 = \Theta_- T_0 g, \)

\[
\Theta_-^\dagger \Theta_+ = T_0^{-1} g T_0,
\quad e^{\theta_0} = B e^{\nu \hat{c}}
\]

In order to introduce highest weight states \( |\lambda_i >, i = 0, 1, \) need to extended the loop to the fully **central extended** Kac-Moody algebra

\[
[h^{(m)}, h^{(n)}] = \hat{c} m \delta_{m+n,0}
\]

\[
[h^{(m)}, E^{(n)}_{\pm \alpha}] = \pm 2 E^{(n+m)}_{\pm \alpha},
\quad [E^{(m)}_{\alpha}, E^{(n)}_{-\alpha}] = h^{(m+n)} + \hat{c} m \delta_{m+n,0},
\]

and introduce \( \nu \) field associated to \( \hat{c}, \) i.e.,

\( B \rightarrow B e^{\nu \hat{c}}, \quad A_0 \rightarrow A_0 + \partial_x \nu \hat{c} \)

such that

\[
< \lambda | B e^{\nu \hat{c}} | \lambda > = < \lambda | T_0^{-1} g T_0 | \lambda > .
\]
The solution for mKdV hierarchy is then given by

\[ e^{-\nu} = \langle \lambda_0 | T_0^{-1} g T_0 | \lambda_0 \rangle \equiv \tau_0, \]
\[ e^{-\phi-\nu} = \langle \lambda_1 | T_0^{-1} g T_0 | \lambda_1 \rangle \equiv \tau_1 \]

and hence, \( \nu = -\partial_x \ln \left( \frac{\tau_0}{\tau_1} \right) \), \( \nu = \partial_x \phi \)

where \( T_0 = e^{xA_{x,vac}} e^{tMA_{tM,vac}} \), \( g = e^{F(\gamma)} \).

Taking \( F(\gamma) \) is an eigenvector (vertex operator) of \( E^{(M)} = A_{tM,vac} \) and \( E^{(1)} = A_{x,vac} \), i.e.,

\[ [E^{(M)}, F(\gamma)] = w_M(\gamma) F(\gamma) . \]

it follows that

\[ T_0^{-1} e^{F(\gamma)} T_0 = e^{\rho(x,t_N;\gamma) F(\gamma)}, \quad \rho(x, t_N; \gamma) = e^{xw_1 + t_N w_N}, \]
We find that the one-soliton solution of the form,

\[
\tau_0 = 1 + C_0 \rho (\gamma, v_0), \quad \tau_1 = 1 + C_1 \rho (\gamma, v_0)
\]

solves all eqns. within the positive mKdV hierarchy for \( w_1 = 2\gamma, \quad w_N = 2\gamma^N \), i.e.,

\[
v = -\partial_x \ln \left( \frac{1 + C_1 \rho}{1 + C_0 \rho} \right).
\]

where

\[
\rho (\gamma, v_0) = \exp \left\{ 2\gamma x + 2\gamma^N t_N \right\}.
\]
The same works for multi-soliton solutions, ie., \( g = \Pi e^{F_i(\gamma_i)} \).

\[ T_0^{-1} \Pi e^{F_i(\gamma_i)} T_0 = \Pi e^{\rho_i(x, t_N; \gamma_i) F_i(\gamma_i)} \]

\( \rho_i(x, t_N) = e^{2\gamma_i x + 2\gamma_i^N t_N} \).

For negative hierarchy \(^2\) and constant vacuum solution

\( \nu = \nu_0 - \partial_x \ln \left( \frac{1 + C_1 \rho}{1 + C_0 \rho} \right) \).

\[ \rho(\gamma, \nu_0) = \exp \left\{ 2\gamma x + \frac{2\gamma t - m}{\nu_0 (\gamma^2 - \nu_0^2)^{m/2}} \right\} \]

\(^2\) JFG, G Starvaggi França, G R de Melo and A H Zimerman, J. of Phys. A42,(2009), 445204
Gauge-Backlund Transformation for Sinh-Gordon

Assume now that two field configurations $\phi_1$ and $\phi_2$ embedded in $A_{x,mKdV}(\phi_1)$ and $A_{x,mKdV}(\phi_2)$ are related by a Backlund gauge transformation, i.e.,

$$K(\phi_1, \phi_2) A_{x,mKdV}(\phi_1) = A_{x,mKdV}(\phi_2) K(\phi_1, \phi_2) + \partial_x K(\phi_1, \phi_2),$$

holds for

$$K(\phi_1, \phi_2) = \begin{bmatrix} 1 & -\frac{\beta}{2\lambda} e^{-(\phi_1 + \phi_2)} \\ -\frac{\beta}{2} e^{(\phi_1 + \phi_2)} & 1 \end{bmatrix}$$

provided Backlund transformation is satisfied, i.e.,

$$\partial_x (\phi_1 - \phi_2) = -\beta \sinh (\phi_1 + \phi_2), \quad \nu_i \equiv \partial_x \phi_i.$$
For the sinh-Gordon, the equations of motion
\[ \partial_{t-1} \partial_x \phi_a = 2 \sinh 2 \phi_a, \quad a = 1, 2 \]
we introduce the time component of the Bäcklund transformation,
\[ \partial_{t-1} (\phi_1 + \phi_2) = \frac{4}{\beta} \sinh (\phi_2 - \phi_1). \] (3)
For higher graded time evolutions the time component of the Backlund transformation can be derived from the appropriated time component of the two dimensional gauge potential.\(^3\) e.g.,

\[ K(\phi_1, \phi_2) A_{t_N, mKdV}(\phi_1) = A_{t_N, mKdV}(\phi_2) K(\phi_1, \phi_2) + \partial_{t_N} K(\phi_1, \phi_2), \]

which for \( t = t_3 \) leads to

\[
\partial_{t_3} \phi_2 \quad - \quad \partial_{t_3} \phi_1 = \frac{\beta}{4} (\partial_x^2 \phi_1 + \partial_x^2 \phi_2) \cosh(\phi_1 + \phi_2) \\
- \quad \frac{\beta}{8} (\partial_x \phi_1 + \partial_x \phi_2)^2 \sinh(\phi_1 + \phi_2) - \frac{\beta^3}{8} \sinh^3(\phi_1 + \phi_2).
\]

Consider now

\[ g_1 = \begin{pmatrix} \zeta & 1 \\ \zeta & -1 \end{pmatrix}, \quad g_2(v, \epsilon) = \begin{pmatrix} 1 & \epsilon \\ -\epsilon v & -v + 2\epsilon \zeta \end{pmatrix}, \quad \zeta^2 = \lambda, \]

which transforms

\[ A_{x,mKdV} = E^{(1)} + v(x, t_N) h = \begin{pmatrix} v & 1 \\ \lambda & -v \end{pmatrix}, \]

into

\[ A_{x,KdV} = g_2 g_1 (A_{x,mKdV}) g_1^{-1} g_2^{-1} - \partial_x g_2 g_2^{-1} = \begin{pmatrix} \zeta & -1 \\ J & -\zeta \end{pmatrix} \]

where \( J = \epsilon \partial_x v - v^2, \quad \epsilon^2 = 1. \)
Following the same line of reasoning propose now

\[ \tilde{K}(J_1, J_2)A_{\mu,KdV}(J_1) = A_{\mu,KdV}(J_2)\tilde{K}(J_1, J_2) + \partial_{\mu}\tilde{K}(J_1, J_2), \]

which can be constructed from \( K \), i.e.,

\[ \tilde{K} = g_2(v_2, \epsilon_2) \left( g_1 K(\phi_1, \phi_2)g_1^{-1} \right) g_2(v_1, \epsilon_1)^{-1} \]

and depend upon \( \epsilon_1, \epsilon_2 \).
For $\epsilon_1 = -\epsilon_2 = \epsilon$ we found

$$\tilde{K}(J_1, J_2, \beta) = -\frac{1}{\zeta} \begin{pmatrix} -\zeta + \frac{1}{2}Q & 1 \\ -\frac{\beta^2}{4} + \frac{1}{4}Q^2 & \zeta + \frac{1}{2}Q \end{pmatrix},$$

where

$$Q = \epsilon(v_1 + v_2) + \frac{\beta}{2}(e^{\phi_1+\phi_2} + e^{-(\phi_1+\phi_2)}) = w_1 - w_2$$

and $J_i = \partial_x w_i$, $i = 1, 2^4$ which generates to the Backlund transformation for the KdV hierarchy

$$J_1 + J_2 = \partial_x P = \frac{\beta^2}{2} - \frac{(w_1 - w_2)^2}{2}, \quad P = w_1 + w_2.$$
Affine Algebraic structure, i.e., $\hat{G}$, $Q$, $E^{(n)}$ provide a systematic method in deriving integrable nonlinear equations,\textit{ Integrable Hierarchies}.

Provide the construction and classification of Soliton Solutions via \textit{Dressing Method}.

How to adapt Dressing method to construct \textit{periodic solutions} (Jacobi Theta functions). where

$$\tau_a = \sum_{k=-\infty}^{+\infty} e^{2\pi i \eta k^2} \rho^k, \quad \eta = \text{deform. parameter}$$

c.f. soliton where

$$\tau_0 = 1 + \rho, \quad \tau_1 = 1 - \rho$$

Provide the Systematic construction of \textit{Backlund Transformation} for higher members of same hierarchy.