Cardy's Ansatz and Left-Right Entanglement Entropy

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Introduction
Space of QFTs

- Conformal field theories: non-trivial renormalization group fixed points of relativistic quantum field theories
- Roughly: massive QFT=RG trajectory/flow=relevant perturbation of a CFT
- Characterizing the massive QFT generally requires non-perturbative methods
- Ground states are of particular interest:
  - Energy density?
  - Phase diagram?
  - Entanglement properties?
  - ...
- This talk: 1+1d, massive perturbations of unitary minimal models

Introduction

What we know about QFT in 1+1d

- Generally: not so much...
- CFT: a lot...
  - OPE+conformal invariance: *n*-point functions
  - Replica trick, twist-fields: real space entanglement entropy
  - ...
- Integrability:
  - Spectrum, finite size effects (TBA)
  - One point functions, form factors: *n*-point functions
  - Twist-field form factors: entanglement entropy
  - ....
- Generally: CFT/form factor perturbation theory

Outline

1. Cardy's Ansatz
   - Motivation
   - The ansatz
2. Left-Right Entanglement Entropy
   - Some context
   - Results for boundary states
3. LREE from Truncated Conformal Space
   - Truncated Hamiltonian approach
   - Results
4. Conclusion, outlook

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The problem

- Consider perturbations of a CFT on a cylinder of circumference L (PBC)

$$H = H_{CFT} + \sum_j \lambda_j \int_0^L dx \Phi_j(x, 0)$$

where the fields Φ_j are relevant, spinless fields of the CFT

- What is the ground state of this Hamiltonian?
- Characterization: boundary states of the CFT?
- Based on Cardy: Bulk RG Flows and Boundary States in CFT (2017)

Cardy's ansatz: motivation
Cardy (2017)

- Imagine that the perturbation is switched on only on the half space
- Interface between the CFT and the massive theory
- CFT side: one sees some boundary condition, boundary RG$\rightarrow$fixed points: conformal boundary conditions
- "...on scales $\sim m^{-1}$ the correlations near the boundary should be those of a conformal boundary condition, deformed by irrelevant boundary operators"

Cardy's ansatz: motivation
Cho, Ludwig, Ryu (2017)

- Entanglement Hamiltonian ($\rho_A = e^{-H_E}$)
- "The entanglement Hamiltonian of the gapped theory is thus the Hamiltonian of a boundary CFT"
- "...we have shown that the low-lying spectrum of the entanglement Hamiltonian of the gapped relativistic field theory is simply the finite size spectrum of the corresponding gapless (conformal) theory with boundary conditions "F" and "B_ϕ"..."
- Supported by numerics

Cardy's ansatz: motivation

Cardy, Calabrese 2006

- Quantum quenches to critical points
- Path integral approach
- Deformed conformal boundary conditions instead of the proper ground state
- Gives the correct behaviour for time dependence of correlation functions

The variational ansatz

The ansatz

$$\left|\{\alpha_a,\tau_\alpha\}\right\rangle = \sum_a \alpha_a e^{-\tau_a H_{CFT}} \left|a\right\rangle$$

where $\{|a\rangle\}$ is a given set of physical boundary states (Cardy states). $\{\alpha_a,\tau_a\}$ are variational parameters to be chosen to minimize the variational energy density in large volume:

$$\lim_{L\to\infty}\frac{1}{L}\frac{\left\langle\{\alpha_a,\tau_\alpha\}\left|H_{CFT}+\sum_j\lambda_j\int\Phi_j\left(x\right)dx\right|\{\alpha_a,\tau_\alpha\}\right\rangle}{\langle\{\alpha_a,\tau_\alpha\}\,|\,\{\alpha_a,\tau_\alpha\}\rangle}$$

Diagonal minimal models

- Cardy states:

$$|a\rangle = \sum_j \frac{S_a^i}{\left(S_0^i\right)^{1/2}} \,|\, i\rangle\rangle$$

where S_a^i are the modular matrix elements and $|\, i\rangle\rangle$ are Ishibashi states

$$|\, i\rangle\rangle = \sum_{N, k_{i,N}} |i, N, k_{i,N}\rangle \otimes \overline{|i, N, k_{i,N}\rangle}$$

- Turns out: the Hamiltonian is diagonal
- The variational ground state energy density

$$E_a = \frac{\pi c}{24\,(2\tau_a)^2} + \sum_{j\neq 0} \frac{S_a^j}{S_a^0} \left(\frac{S_0^0}{S_0^j}\right)^{1/2} \frac{\pi^{\Delta_j}}{(4\tau_a)^{\Delta_j}}$$

Integrable Examples

E_8, E_7, E_6 scattering theories: $c = 1/2 + \Phi_{1/16,1/16}$, $c = 7/10 + \Phi_{1/10,1/10}$, $c = 6/7 + \Phi_{1/7,1/7}$
Mass coupling relation, bulk energy constant:

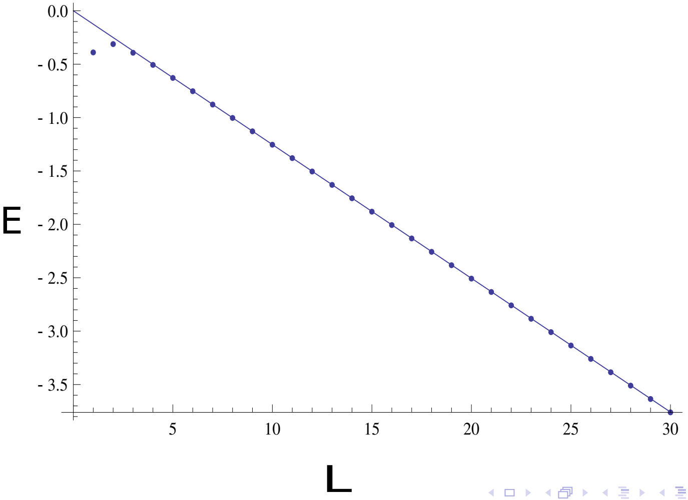
$$\lambda_j = \kappa_j m_l$$
$$b_j = b_j\left(\kappa_j, m_l\right)$$

are known from Fateev 1994. For the bulk energy constants one has:

$m_l = 1$	Exact	Cardy's ansatz
E_8	-0.0617286	-0.0615441
E_7	-0.0942097	-0.0935744
E_6	-0.105662	-0.10419

A non-integrable example

$c = 7/10 + \Phi_{3/80,3/80}$, Cardy's ansatz vs. TCSA



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Topological entanglement entropy
Li, Haldane 2008, Qi, Katsura, Ludwig 2012

- Topological quantum states in 2+1 possess edge states described by 1+1 d chiral CFT
- The reduced density matrix of a spatial region of the gapped model: thermal density matrix of a chiral CFT on the cut
- Possible interpretation: left movers on the upper cut, right movers on the lower
- The topological entanglement entropy is the entanglement entropy between left and right movers.

General result for boundary states
Das, Datta (2015)

- Consider a regularized boundary state

$$|\mathcal{B}\rangle = \frac{e^{-\tau H_{CFT}}}{(\mathcal{N}_B)^{1/2}}\,|B\rangle$$

where $|B\rangle$ is a general linear combination of Ishibashi states

$$|B\rangle = \sum_j \psi_B^j \,|j\rangle\rangle$$

and $\mathcal{N}_B$ is a normalization factor given in terms of Virasoro characters:

$$\mathcal{N}_B = \sum_j \left|\psi_B^j\right|^2 \chi_j\left(e^{-8\pi(\tau/L)}\right)$$

Result for boundary states

- Reduced density matrix (say left)

$$\rho_L^{(\mathcal{B})} = \frac{1}{\mathcal{N}_B} \mathrm{Tr}_R \left(e^{-\tau H_{CFT}} \, |B\rangle\langle B| \, e^{-\tau H_{CFT}} \right)$$
$$= \frac{1}{\mathcal{N}_B} \sum_{a,N,k_{a,N}} |\psi_B^a|^2 \, e^{-8\pi\tau/L(h_a+N-c/24)} \, |a, N, k_{a,N}\rangle\langle a, N, k_{a,N}|$$

where the sum runs over primaries (a), levels (N) and states at a given level in a given module ($k_{a,N}$)

- Using modular invariance and $L/\tau \gg 1$ expansion leads to

$$S_{\mathcal{B}}^{LR} = \frac{\pi c L}{24\tau} - \frac{\sum_a S_0^a \, |\psi_B^a|^2 \log\left(|\psi_B^a|^2\right)}{\sum_a S_0^a \, |\psi_B^a|^2} + \log\left(\sum_a S_0^a \, |\psi_B^a|^2\right)$$

Result for boundary states

- In particular for Cardy states in diagonal models:

$$|B\rangle = |a\rangle = \sum_j \frac{S_a^j}{\left(S_0^j\right)^{1/2}} |j\rangle\rangle$$

$$S_{|a\rangle}^{LR} = \frac{\pi c L}{24\tau} - \sum_j \left(S_a^j\right)^2 \log\left(\frac{\left(S_a^j\right)^2}{S_a^0}\right)$$

- Note that Cardy's ansatz for bulk ground state is of this type with some τ given by the minimization!!!
- How to get access to this?

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Truncated Hamiltonian Approach

- Yurov, Zamolodchikov 1990
- Truncated Hamiltonian/Space/Conformal Space/Fermionic Space Approach
- Consider the Hamiltonian

$$H = H_0 + \lambda V$$

  acting on some infinite dimensional, but discrete Hilbert space
- Spectrum is H_0 is well-known $(\{|n\rangle, E_n, \langle m|O|n\rangle\})$
- Take the projector P_Λ to low a energy subspace with $E < \Lambda$, write the (finite dim.) matrix of

$$P_\Lambda H P_\Lambda$$

  and diagonalize it numerically.
- Low energy spectrum

Truncated Hamiltonian Approach

- Spectrum, scattering phases
- Form factors
- Correlation functions
- Quantum quenches, time evolution
- Real-space second Rényi entropy
- RG improvements
- Higher dimension
- ....
- For a review: James, Konik, Lecheminant, Robinson, Tsvelik 2017

Truncated Conformal Space Approach

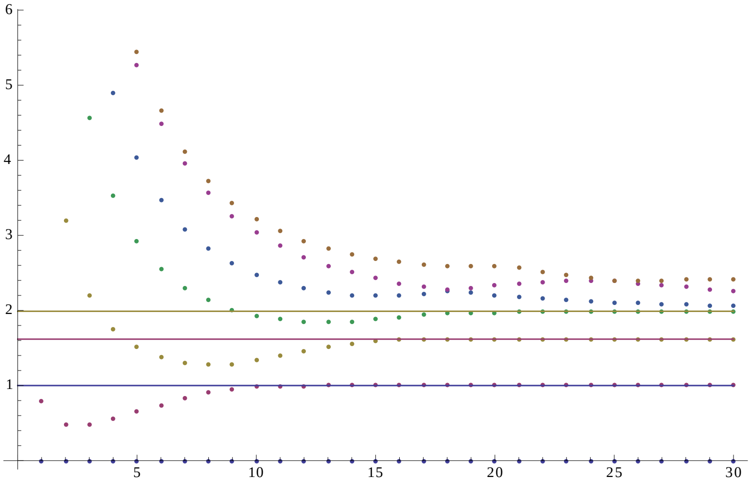
- CFT Hilbert space (diagonal mod. inv.)

$$\mathcal{H}_{CFT} = \bigoplus_{h \in \mathrm{K.t.}} \mathcal{V}_h^L \otimes \mathcal{V}_h^R \subset \mathcal{H}_L \otimes \mathcal{H}_R$$

- The space of states can be generated, matrix elements can be calculated using Ward identities
- Periodic boundary conditions, spinless perturbations: different total momentum sectors can be treated separately
- Convergence depends on the perturbation: the more relevant, the more convergent

TCSA, an example spectrum: E_8

Relative energies against the system size



Ground state from TCSA

- The normalized ground state given by TCSA is a pure state wrt. left-right decomposition:

$$|\Psi\rangle = \sum_{l \in \mathcal{H}_L, r \in \mathcal{H}_R} C_{lr} \, |l\rangle \otimes |r\rangle$$

where the only nonzero coefficients come from states

$$C_{lr} \sim \delta_{h_l, h_r} \delta_{N_l, N_r}$$

- Density matrix elements can be easily calculated:

$$\rho_{l,r;l',r'} = C^*_{l,r} C_{l',r'}$$

- Trace over right-movers can be carried out, and the reduced density matrix is block diagonal (orthogonality of the modules and zero momentum condition)

Cardy's ansatz, LREE: reminder

- Cardy's ansatz

$$|\Psi\rangle \sim e^{-\tau H_{CFT}} |a\rangle$$

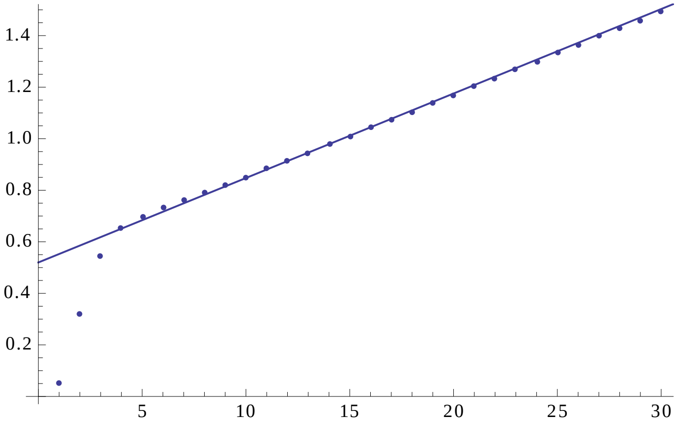
- Result for LREE

$$|a\rangle_{reg} = \frac{e^{-\tau H_{CFT}}}{\mathcal{N}_B} |a\rangle$$

$$S^{LR}_{|a\rangle} = \frac{\pi c L}{24\tau} - \sum_j \left(S^j_a\right)^2 \log\left(\frac{\left(S^j_a\right)^2}{S^0_a}\right)$$

LREE from TCSA: E_8

GS LREE volume dependence: TCSA vs. Cardy's Ansatz+Das, Datta



LREE from TCSA: Tricritical Ising+$\Phi_{3/80,3/80}$

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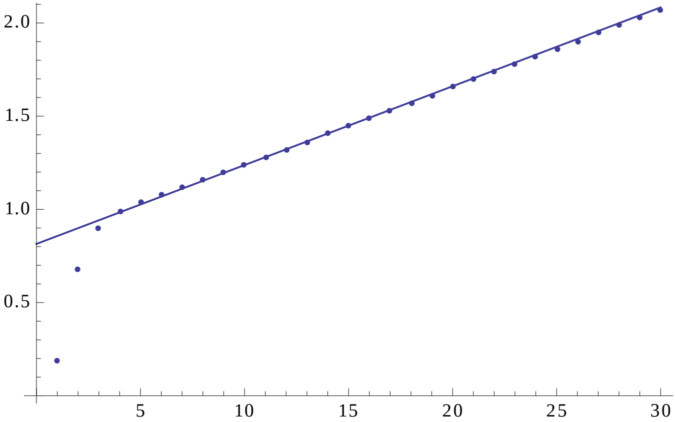


Left-right entanglement spectrum

- Reminder: the left reduced density matrix:

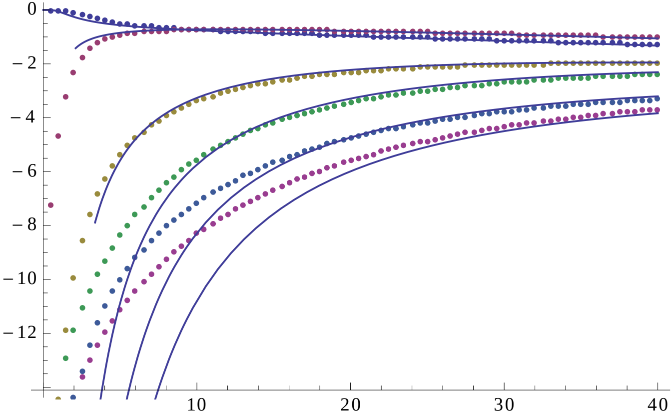
$$\rho_L^{(\mathcal{B})} = \frac{1}{\mathcal{N}_B} \sum_{a,N,k} |\psi_B^a|^2 \, e^{-8\pi\tau/L(h_a+N-c/24)} \, |a,N,k\rangle\langle a,N,k|$$

- One can read off the entanglement spectrum

$$2\log|\psi_B^a| - \log\mathcal{N}_B - 8\pi\frac{\tau}{L}\left(h_a + N - \frac{c}{24}\right)$$

Left-right entanglement spectrum: E_8

Left-right entanglement spectrum: TCSA vs. Cardy's Ansatz+Das, Datta



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- Cardy's ansatz captures the energy density and the left right entanglement entropy of ground states of massive QFT
- Left-right entanglement spectrum matches
- New application of TCSA

- Less relevant perturbations: less impressive
  - Higher TCSA cut-offs, TCSA RG implementation
  - Is the ansatz getting worse? How to fix?
- Time evolution?
- Alternative calculations?

Thank you!

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