The XYZ spin-chain, the eight-vertex model and supersymmetry

Exactly solvable quantum chains – Natal, 29/06/2018

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Outline

1. The XYZ spin-chain
2. Supersymmetry
3. The eight-vertex model
4. Resut and sketch of the proof
   - Result
   - Susy singlets - XYZ ground states
   - Susy and integrability - TM eigenvalue
5. Open case
6. Conclusion

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Notation

Hilbert space:

Site: $V_i = \mathbb{C}^2 = \text{span}(|\uparrow\rangle, |\downarrow\rangle)$ with $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Chain of L sites: $V^L = V_1 \otimes V_2 \otimes \cdots \otimes V_L$

Configuration: canonical basis of V^L

$$|s_1 s_2 \ldots s_L\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_L\rangle, \quad |s_i\rangle = |\uparrow\rangle, |\downarrow\rangle$$

XYZ spin-chain Hamiltonian

XYZ Hamiltonian with periodic b.c.

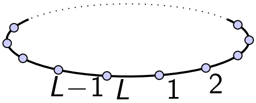
$$H_{XYZ} = \sum_{j=1}^{L} h_{j,j+1}$$

with anisotropy parameters J_x, J_y and J_z

$$h_{j,j+1} = \frac{-1}{2}\left(J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z\right)$$

The σ_i^α is the Pauli matrix σ^α acting on V_i,

$$\sigma_{L+1}^\alpha = \sigma_1^\alpha, \quad \alpha = x, y, z.$$



Spectrum: $E_0 < E_1 < E_2 < \cdots$

Large L limit

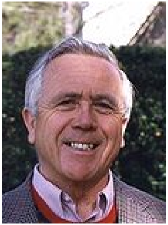
Theorem

If the anistropy parameters obey the relation

$$J_x J_y + J_x J_z + J_y J_z = 0, \quad J_x + J_y + J_z > 0$$

then the ground-state energy per site satisfies

$$\lim_{L\to\infty} \frac{E_0}{L} = -\frac{1}{2}(J_x + J_y + J_z).$$



"[...] which is a remarkably simple result!"

[Baxter '72]

For finite chains

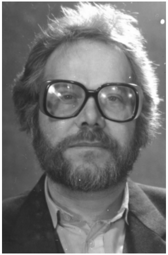
Conjecture

If the anisotropy parameters obey the relation

$$J_x J_y + J_x J_z + J_y J_z = 0, \quad J_x + J_y + J_z > 0$$

and $L = 2n + 1$ is odd, then the ground-state energy is

$$E_0 = -\frac{1}{2}(2n+1)(J_x + J_y + J_z).$$



The importance of being odd !

[Stroganov '01, Razumov & Stroganov '10]

Why is $J_xJ_y + J_xJ_z + J_yJ_z = 0$ interesting ?

XXZ limit: $J_x = J_y = 1$, $J_z = -1/2$

- XXZ spin-chain at $\Delta = -1/2$
- The finite-size ground-state is related to enumerative combinatorics
- Exact finite-size correlation functions

"The values $\Delta = \pm 1/2$ are truly exceptional"

[de Gier, Batchelor, Nienhuis, di Fransesco, Zinn-Justin, Cantini...]

XYZ case

- Ground states $\leftrightarrow$ Painlevé *VI* equation
- Combinatorics
- Supersymmetry

[Bazhanov, Mangazeev, Rosengren, Zinn-Justin, Fendley, Hagendorf...]

XYZ
model

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Supersymmetry
$\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = H$

Supersymmetry

$\mathcal{N} = 2$ supersymmetric quantum mechanics:

Supercharges: $\mathfrak{Q}$, $\mathfrak{Q}^\dagger$

verifying

$$\mathfrak{Q}^2 = \left(\mathfrak{Q}^\dagger\right)^2 = 0, \quad \{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = H.$$

Properties:

- non-negative energy ($E \geqslant 0$),
- zero-energy state ($H|\psi\rangle = 0$) verifies

$$\mathfrak{Q}|\psi\rangle = 0, \quad \mathfrak{Q}^\dagger|\psi\rangle = 0.$$

  Zero-energy states are called supersymmetry singlets.
- A zero-energy state is in $\ker(\mathfrak{Q})$ but not in $\mathrm{im}(\mathfrak{Q})$.

Supersymmetry II.

Quotient space

$$\mathcal{H}(\mathfrak{Q}) = \frac{\ker\ (\mathfrak{Q})}{\mathrm{im}\ (\mathfrak{Q})}$$

Elements of $\mathcal{H}(\mathfrak{Q})$ are equivalence classes of vectors in $\ker\{\mathfrak{Q}\}$.
Representatives of the classes: $[|\phi\rangle] = [|\phi\rangle + \mathfrak{Q}|\alpha\rangle]$.

$$\mathcal{H}(\mathfrak{Q}^\dagger) = \frac{\ker\ (\mathfrak{Q}^\dagger)}{\mathrm{im}\ (\mathfrak{Q}^\dagger)}$$

Representatives of the classes: $[|\phi\rangle] = [|\phi\rangle + \mathfrak{Q}^\dagger|\beta\rangle]$.

Supersymmetry III.

Theorem

The space of solutions to

$$\mathfrak{Q}|\psi\rangle = 0, \quad \mathfrak{Q}^\dagger|\psi\rangle = 0$$

is isomorphic to $\mathcal{H}(\mathfrak{Q}) \simeq \mathcal{H}(\mathfrak{Q}^\dagger)$.

- The degeneracy of $E = 0$ is $\dim\, \mathcal{H}(\mathfrak{Q})$.
- If $|\psi\rangle$ is the representative of a non-zero element of $\mathcal{H}(\mathfrak{Q})$, then there exists (a non unique) $|\alpha\rangle$ such that

$$|\psi_0\rangle = |\psi\rangle + \mathfrak{Q}|\alpha\rangle$$

is a zero-energy state (proof: Hodge decomposition).

[Witten, '82]

XYZ
model

Supersymmetry
$\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = H$

XYZ and the supersymmetry I.

Local supercharge $\mathfrak{q} : V \to V^2$

$$\mathfrak{q}|\uparrow\rangle = 0, \quad \mathfrak{q}|\downarrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle - \zeta|\downarrow\rangle \otimes |\downarrow\rangle$$

Let $\mathcal{S}$ be the translation operator acting on V^L:

$$\mathcal{S}|s_1 s_2 \dots s_L\rangle = |s_L s_1 \dots s_{L-1}\rangle,$$

we define $\mathfrak{q}_1$ as $\mathfrak{q}$ acting on the first site

$$\mathfrak{q}_1 = \mathfrak{q} \otimes 1 \otimes \cdots \otimes 1$$

and $\mathfrak{q}_j$ for $j = 0, \dots, L$ by

$$\mathfrak{q}_j = \mathcal{S}^{j-1}\mathfrak{q}_1\mathcal{S}^{1-j}.$$

XYZ and the supersymmetry II.

Construction of the supercharge

1. On W^L, the subspace of alternate-cyclic states,

$$W^L = \{|\psi\rangle \in V^L | \mathcal{S}|\psi\rangle = (-1)^{L+1}|\psi\rangle\}$$

$$\mathfrak{Q} = \sqrt{\frac{L}{L+1}} \sum_{j=0}^{L} (-1)^j \mathfrak{q}_j$$

2. On $V^L \backslash W^L$, $\mathfrak{Q} = 0$

Its adjoint is $\mathfrak{Q}^\dagger$: $\langle\psi|\mathfrak{Q}^\dagger|\phi\rangle = (\langle\phi|\mathfrak{Q}|\psi\rangle)^*$.

The supercharges $\mathfrak{Q}$ and $\mathfrak{Q}^\dagger$ are length-changing operators

$$\mathfrak{Q} : V^L \to V^{L+1}, \quad \mathfrak{Q}^\dagger : V^L \to V^{L-1},$$

$\mathfrak{Q}$ and $\mathfrak{Q}^\dagger$ maps W^L onto W^{L+1} and W^{L-1}, respectively.

XYZ and the supersymmetry III.

The supercharge is nilpotent

The supercharge and its adjoint satisfy

$$\mathfrak{Q}^2 = 0, \quad (\mathfrak{Q}^\dagger)^2 = 0.$$

The XYZ Hamiltonian is supersymmetric on W^L

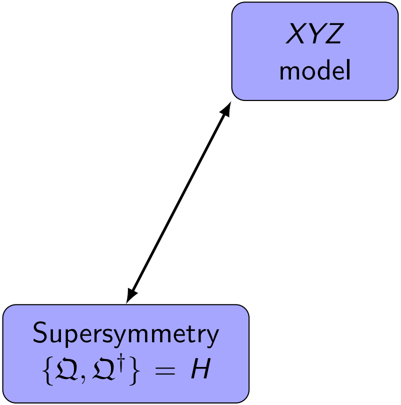
$$\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \begin{cases} H_{XYZ} + L(3+\zeta^2)/4 & \text{on } W^L \\ 0 & \text{on } V^L \backslash W^L \end{cases}$$

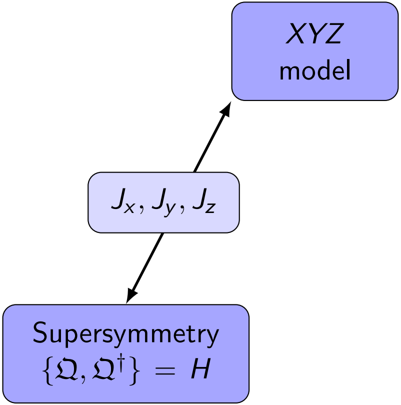
with

$$J_x = 1+\zeta, \; J_y = 1-\zeta, \; J_z = \frac{1}{2}(\zeta^2 - 1). \quad (\star)$$

$$(\star) \Leftrightarrow J_xJ_y + J_xJ_z + J_yJ_z = 0, \quad J_x + J_y + J_z > 0.$$

[Fendley & Yang '04], [Fendley & Hagendorf '10, '12]


XYZ
model
Supersymmetry
$\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = H$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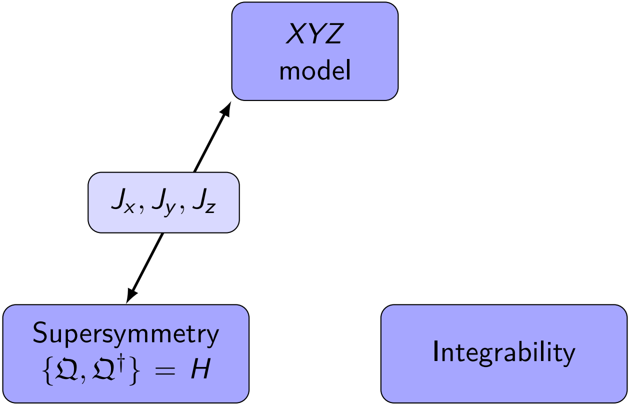
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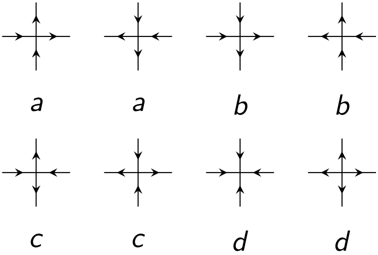
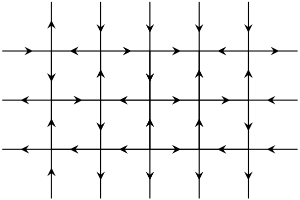
Integrability

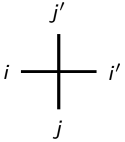
The eight-vertex model

- Square lattice with $L = 2n + 1$ vertical lines
- All edges are oriented
- Generalised ice-rule at each vertex
- Periodic boundary conditions along the horizontal direction





The R-matrix and the transfer matrix

The R-matrix encodes the weigths: 

$:= \langle i', j' | R | i, j \rangle$

As a matrix, in the basis of $V \otimes V$, we have

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}.$$

Transfer matrix for L vertical lines and periodic b.c.

$$\mathcal{T} = \mathrm{tr}_0 \left(R_{0L} R_{0L-1} \cdots R_{01} \right)$$

Here, R_{0j} is the R-matrix acting on V_0 and V_j in $V_0 \otimes V^L$.

Transfer matrix

Commutation relation between H_{XYZ} and $\mathcal{T}$

$[H_{XYZ}, \mathcal{T}] = 0$ provided that

$$J_x = 1 + \zeta, \quad J_y = 1 - \zeta, \quad J_z = \frac{a^2 + b^2 - c^2 - d^2}{2ab}, \quad \zeta = \frac{cd}{ab}$$

[Sutherland '70]

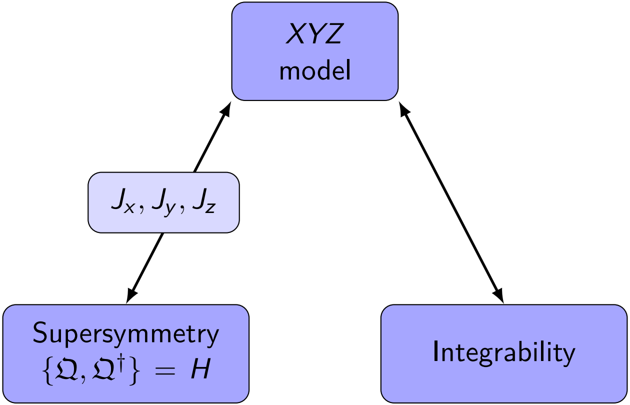
For the special case $J_x J_y + J_x J_z + J_y J_z = 0$, we find the

supersymmetric eight-vertex model

The vertex weights are related by

$$(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab).$$

We condider the case $a, b, c, d \neq 0$


XYZ model
J_x, J_y, J_z
Supersymmetry $\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = H$
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Special eigenvalue of the transfer matrix

Theorem [C.Hagendorf, J.L.]

For $L = 2n + 1,\ n \geqslant 1$, the transfer matrix of the supersymmetric eight-vertex model possesses the doubly degenerate eigenvalue

$$\Theta_n = (a + b)^{2n+1}.$$

Its eigenspace is spanned by the ground states of H_{XYZ} with anisotropy parameters

$$J_x = 1 + \zeta, \quad J_y = 1 - \zeta, \quad J_z = \frac{1}{2}(\zeta^2 - 1), \quad \zeta = \frac{cd}{ab} \neq 0$$

and the ground state eigenvalue $E_0 = -\frac{1}{4}(2n + 1)(3 + \zeta^2)$.
Moreover, if $a, b, c, d > 0$, then Θ_n is the largest eigenvalue.

The strategy of the proof

Susy singlets
XYZ ground states

Susy and integrability
TM eigenvalue

The strategy of the proof

Susy singlets
XYZ ground states

$\downarrow$

Susy and integrability
TM eigenvalue

About the proof:

- Elementary !
- Based on Susy.
- No need of the explicit g.s.
- $d = 0 \rightarrow XXZ$

How can supersymmetry characterise XYZ ground states?

(Co)homology

Solutions to $\mathfrak{Q}|\psi\rangle = 0, \mathfrak{Q}^\dagger|\psi\rangle = 0 \leftrightarrow$ equivalence classes of $\mathcal{H}$

$$W^1 \underset{\mathfrak{Q}^\dagger}{\overset{\mathfrak{Q}}{\rightleftharpoons}} W^2 \underset{\mathfrak{Q}^\dagger}{\overset{\mathfrak{Q}}{\rightleftharpoons}} \cdots \underset{\mathfrak{Q}^\dagger}{\overset{\mathfrak{Q}}{\rightleftharpoons}} W^L \underset{\mathfrak{Q}^\dagger}{\overset{\mathfrak{Q}}{\rightleftharpoons}} W^{L+1} \underset{\mathfrak{Q}^\dagger}{\overset{\mathfrak{Q}}{\rightleftharpoons}} \cdots$$

Cohomology sequence :

$$\mathcal{H}^L(\zeta) = \frac{\ker\{\mathfrak{Q} : W^L \to W^{L+1}\}}{\operatorname{im}\{\mathfrak{Q} : W^{L-1} \to W^L\}}, \quad L \geqslant 1$$

Homology sequence :

$$\mathcal{H}_L(\zeta) = \frac{\ker\{\mathfrak{Q}^\dagger : W^L \to W^{L-1}\}}{\operatorname{im}\{\mathfrak{Q}^\dagger : W^{L+1} \to W^L\}}, \quad L \geqslant 1$$

Proposition

For each $n \geqslant 1$ and $\zeta \neq 0$, we have

$$\mathcal{H}^{2n}(\zeta) = 0, \quad \mathcal{H}^{2n+1}(\zeta) = \mathbb{C}[|\phi_n(\zeta)\rangle] \oplus \mathbb{C}[|\bar{\phi}_n(\zeta)\rangle],$$
$$\mathcal{H}_{2n}(\zeta) = 0, \quad \mathcal{H}_{2n+1}(\zeta) = \mathbb{C}[|\phi_n(\zeta^{-1})\rangle] \oplus \mathbb{C}[|\bar{\phi}_n(\zeta^{-1})\rangle],$$

with

$$|\phi_n(\zeta)\rangle = \sum_{m=0}^{n} \zeta^{n-m} \sum_{1\leqslant x_1<\cdots<x_{2m}\leqslant 2n+1} \sigma_{x_1}^{+}\cdots\sigma_{x_{2m}}^{+}|{\downarrow\downarrow}\cdots{\downarrow}\rangle$$

and the spin-reversal image

$$|\bar{\phi}_n(\zeta)\rangle = \zeta^n \mathcal{R}|\phi_n(\zeta^{-1})\rangle, \quad \mathcal{R} = \sigma_1^x\cdots\sigma_{2n+1}^x.$$

idea: compute $\mathcal{H}^L(1)$ and use a conjugaison argument.

Supersymmetry singlets

Non-trivial representatives $\rightarrow$ supersymmetry singlets in W^L

Theorem

Let $\zeta \neq 0$ and $n \geqslant 1$. If $L = 2n$, then $H_{XYZ}|_{W^{2n}} > E_0$.
If $L = 2n+1$, then the space of the ground states of $H_{XYZ}|_{W^{2n+1}}$ is spanned by the supersymmetry singlets

$$|\Psi_n\rangle = |\phi_n(\zeta)\rangle + \mathfrak{Q}|\alpha_n\rangle = \mu_n|\phi_n(\zeta^{-1})\rangle + \mathfrak{Q}^\dagger|\beta_n\rangle$$
$$|\bar{\Psi}_n\rangle = |\bar{\phi}_n(\zeta)\rangle + \mathfrak{Q}|\bar{\alpha}_n\rangle = \bar{\mu}_n|\bar{\phi}_n(\zeta^{-1})\rangle + \mathfrak{Q}^\dagger|\bar{\beta}_n\rangle,$$

with $|\alpha_n\rangle, |\bar{\alpha}_n\rangle \in W^{2n}$, $|\beta_n\rangle, |\bar{\beta}_n\rangle \in W^{2n+2}$ and $\mu_n \neq 0, \bar{\mu}_n \neq 0$.

Example for $L = 3$

We have

$$|\phi_1(\zeta)\rangle = \zeta|\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle$$
$$|\bar{\phi}_1(\zeta)\rangle = |\uparrow\uparrow\uparrow\rangle + \zeta(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle).$$

The following alternate-cyclic states

$$\begin{aligned}|\Psi_1\rangle &= |\phi_1(\zeta)\rangle = \zeta|\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle \\ |\bar{\Psi}_1\rangle &= |\bar{\phi}_1(\zeta)\rangle + \sqrt{\frac{3}{2}}\left(\frac{\zeta^2-1}{\zeta^2+3}\right)\mathfrak{Q}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ &= \frac{4\zeta}{\zeta^2+3}\left(\zeta|\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)\end{aligned}$$

are eigenstates of H_{XYZ} with eigenvalue $E_0 = -\frac{3}{4}(3+\zeta^2)$.

Properties of $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$

- The representatives are not uniques !

$$
\begin{aligned}
|\bar{\Psi}_1\rangle &= 4\zeta|\uparrow\uparrow\uparrow\rangle - \sqrt{\frac{3}{2}}\frac{4\zeta}{3+\zeta^2}\mathfrak{Q}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\
&= \frac{4\zeta}{3+\zeta^2}\left(\zeta|\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)
\end{aligned}
$$

- spin parity: $\mathcal{P} = (-1)\sigma_1^z\sigma_2^z\cdots\sigma_L^z$ measures the parity of the number of spins up

$$
\mathcal{P}|\Psi_n\rangle = |\Psi_n\rangle, \quad \mathcal{P}|\bar{\Psi}_n\rangle = -|\bar{\Psi}_n\rangle
$$

- spin reversal operator: $\mathcal{R} = \sigma_1^x\sigma_2^x\cdots\sigma_L^x$

$$
\mathcal{R}|\Psi_n\rangle \propto |\bar{\Psi}_n\rangle
$$

So far, we focused on W^{2n+1} ... But what can we say about the ground state of H_{XYZ} on V^{2n+1}?

XYZ ground state

Theorem

For each $n \geqslant 1$ and $\zeta \neq 0$, the states $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$ span the space of the ground states of H_{XYZ} on V^{2n+1}

Strategy: we prove that the XYZ ground states are in W^{2n+1}
Sketch of the proof:

1. It is sufficient to prove the statement for $\zeta > 0$.
2. There exists a constant μ s.t. $\mu - H_{XYZ}|_{\mathcal{P}\equiv\pm1}$ is non negative and irreducible. We apply the Perron-Frobenius theorem. The Perron states $|\psi_\pm\rangle$ are the unique ground states of $H_{XYZ}|_{\mathcal{P}\equiv\pm1}$ with real components.
3. By unicity, $\mathcal{S}|\psi_\pm\rangle = t_\pm|\psi_\pm\rangle$, $t_\pm^{2n+1} = 1$. Hence $t_\pm = 1$.

[Yang & Yang '66]

How can supersymmetry help to compute the action of the transfer matrix on the XYZ ground states?

Commutation relation between R and $\mathfrak{q}$

We introduce a length-increasing operator A:

$$A|\uparrow\rangle = d\left(-\frac{c}{a}|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \quad A|\downarrow\rangle = c\left(|\uparrow\uparrow\rangle - \frac{d}{b}|\downarrow\downarrow\rangle\right).$$

We define $A_0^j : V_0 \otimes V^L \to V_0 \otimes V^{L+1}$ by

$$A_0^1 = A \otimes 1 \otimes \cdots \otimes 1, \quad A_0^j = \mathcal{S}^{j-1} A_0^1 \mathcal{S}^{1-j}, \quad j = 1, \ldots, L.$$

Local commutation relation between R and $\mathfrak{q}$

$$R_{0i+1} R_{0i} (1 \otimes \mathfrak{q}_i) + (a+b)(1 \otimes \mathfrak{q}_i) R_{0i} = A_0^{i+1} R_{0i} + R_{0i+1} A_0^i$$

if $(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab)$.

The RHS is a local boundary term !

The transfer matrix preserves the supersymmetry

The local commutation relation leads to the

commutation relation between $\mathcal{T}$ and $\mathfrak{Q}$ on V^L

$$\mathcal{T}\mathfrak{Q} = -(a+b)\mathfrak{Q}\mathcal{T}$$

if $(a^2+ab)(b^2+ab) = (c^2+ab)(d^2+ab)$.

proof:

- trivial on $V^L \backslash W^L$,
- on W^L, the alternate sum $\mathfrak{Q} = \sum_j (-1)^j \mathfrak{q}_j$ cancels the local boundary terms.

Matrix elements of $\mathcal{T}$

Matrix elements w.r.t susy singlets $\longrightarrow$ Matrix elements w.r.t representatives

Let $|\Psi\rangle$ be a supersymmetry singlet with decompositions

$$|\Psi\rangle = |\phi\rangle + \mathfrak{Q}|\alpha\rangle, \quad |\Psi\rangle = |\phi'\rangle + \mathfrak{Q}^\dagger|\beta\rangle$$

then we have the expectation value

$$\langle\Psi|\mathcal{T}|\Psi\rangle = \langle\phi'|\mathcal{T}|\phi\rangle$$

The proof is simple:

$$\begin{aligned}\langle\Psi|\mathcal{T}|\Psi\rangle &= \langle\phi'|\mathcal{T}|\Psi\rangle + \cancel{\langle\beta|\mathfrak{Q}\mathcal{T}|\Psi\rangle} \\ &= \langle\phi'|\mathcal{T}|\phi\rangle + \cancel{\langle\phi'|\mathcal{T}\mathfrak{Q}|\alpha\rangle}\end{aligned}$$

Eigenvectors of $\mathcal{T}$ – I.

For $L = 2n + 1$, $\Theta_n = (a + b)^{2n+1}$

Θ_n-eigenstates $\Rightarrow$ XYZ ground states !

Indeed, if $|\Psi\rangle$ satisfies $\mathcal{T}|\Psi\rangle = \Theta_n|\Psi\rangle$ then

$$\mathcal{S}|\Psi\rangle = |\Psi\rangle, \quad H_{XYZ}|\Psi\rangle = -\frac{1}{4}(2n+1)(\zeta^2+3)|\Psi\rangle, \quad \zeta = \frac{cd}{ab}$$

if $(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab)$.

XYZ ground states $\Rightarrow$ Θ_n-eigenstates ?

Eigenvectors of $\mathcal{T}$ – II.

XYZ ground states $\Rightarrow$ Θ_n-eigenstates ?

Yes ! As $[\mathcal{T}, H_{XYZ}] = 0$ and $[\mathcal{T}, \mathcal{P}] = 0$, one has

$$\langle \bar{\Psi}_n | \mathcal{T} | \Psi_n \rangle = \langle \Psi_n | \mathcal{T} | \bar{\Psi}_n \rangle = 0$$

Hence $|\bar{\Psi}_n\rangle$ and $|\Psi_n\rangle$ are eigenstates of $\mathcal{T}$.

$\rightarrow$ Last step: compute the eigenvalue

Eigenvalue of $\mathcal{T}$ – I.

The eigenvalue of $\mathcal{T}$ w.r.t. $|\Psi_n\rangle$, $|\bar{\Psi}_n\rangle$ are the diagonal elements

$$\Theta_n = \frac{\langle \Psi_n|\mathcal{T}|\Psi_n\rangle}{\langle \Psi_n|\Psi_n\rangle} = \frac{\langle \bar{\Psi}_n|\mathcal{T}|\bar{\Psi}_n\rangle}{\langle \bar{\Psi}_n|\bar{\Psi}_n\rangle}$$

...that we express in terms of matrix elements w.r.t. representatives

$$\Theta_n = \langle \bar{\phi}_n(\zeta^{-1})|\mathcal{T}|\uparrow\uparrow\cdots\uparrow\rangle, \quad \zeta = cd/ab.$$

Example: L=3

Matrix element for $n=1$, $\zeta = cd/ab$

$$\begin{aligned}\Theta_1 &= \langle\uparrow\uparrow\uparrow|\mathcal{T}|\uparrow\uparrow\uparrow\rangle + \tfrac{1}{\zeta}\big(\langle\uparrow\downarrow\downarrow|\mathcal{T}|\uparrow\uparrow\uparrow\rangle + \langle\downarrow\uparrow\downarrow|\mathcal{T}|\uparrow\uparrow\uparrow\rangle + \langle\downarrow\downarrow\uparrow|\mathcal{T}|\uparrow\uparrow\uparrow\rangle\big)\\ &= (a^3+b^3) \qquad + 3\zeta^{-1}cd(a+b) = (a+b)^3\end{aligned}$$

Eigenvalue of $\mathcal{T}$ – II.

Proposition

For each $n \geqslant 1$,

$$\Theta_n = \langle \bar{\phi}_n(\zeta^{-1})|\mathcal{T}|\uparrow\uparrow\cdots\uparrow\rangle = (a+b)^{2n+1}$$

The proof is elementary combinatorics. It relies on a mapping from the terms in the sum

$$\sum_{m=0}^{n} \sum_{1\leqslant x_1<\cdots<x_{2m}\leqslant 2n+1} \zeta^{-m}\langle\uparrow\uparrow\cdots\uparrow|\sigma^+_{x_1}\cdots\sigma^+_{x_{2m}}\mathcal{T}|\uparrow\uparrow\cdots\uparrow\rangle$$

and the set Γ of words $\gamma = (\gamma_1,\dots,\gamma_{2n+1})$ with letters $\gamma_j \in \{a,b\}$. Proviging to each word the weight $\omega(\gamma) = \gamma_1\gamma_2\cdots\gamma_{2n=1}$, the sum reduces to

$$\Theta_n = \sum_{\Gamma}\omega(\gamma) = (a+b)^{2n+1}$$

Special eigenvalue of the transfer matrix

Theorem [C.Hagendorf, J.L.]

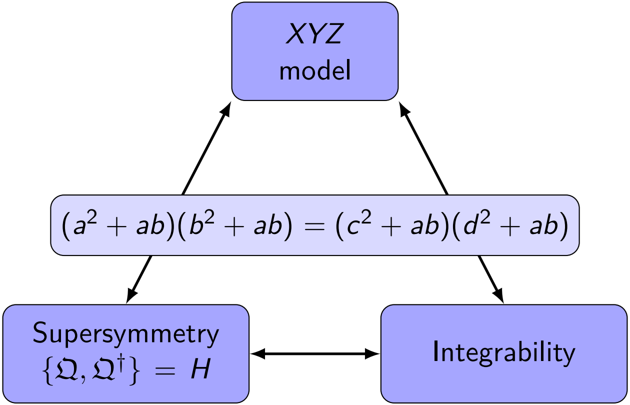
For $L = 2n+1,\ n \geqslant 1$, the transfer matrix of the supersymmetric eight-vertex model possesses the doubly degenerate eigenvalue

$$\Theta_n = (a+b)^{2n+1}$$

Its eigenspace is spanned by the ground states of H_{XYZ} with anisotropy parameters

$$J_x = 1+\zeta, \quad J_y = 1-\zeta, \quad J_z = \frac{1}{2}(\zeta^2-1), \quad \zeta = \frac{cd}{ab} \neq 0$$

and the ground state eigenvalue $E_0 = -\frac{1}{4}(2n+1)(3+\zeta^2)$.
Moreover, if $a, b, c, d > 0$, then Θ_n is the largest eigenvalue.


XYZ model
$(a^2+ab)(b^2+ab) = (c^2+ab)(d^2+ab)$
Supersymmetry $\{\mathfrak{Q},\mathfrak{Q}^\dagger\} = H$
Integrability

Outline

1. The XYZ spin-chain
2. Supersymmetry
3. The eight-vertex model
4. Resut and sketch of the proof
   - Result
   - Susy singlets - XYZ ground states
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5. Open case
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Hamiltonian with open boundary conditions

Open chain with boundary magnetic fields

$$H_{XYZ} = -\frac{1}{2}\sum_{j=1}^{L-1}\left(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z\right)+(h_B)_1+(h_B)_L$$

with the boundary terms

$$h_B = \lambda_0 + \lambda_x\sigma^x + \lambda_y\sigma^y + \lambda_z\sigma^z$$

Can we choose the λ_x, λ_y and λ_z such that the Hamiltonian is supersymmetric with

$$J_x = 1+\zeta, \quad J_y = 1-\zeta, \quad J_z = \frac{1}{2}(\zeta^2-1) \quad ?$$

Open chain and Susy

The XYZ local supercharge is not unique. One can consider its image under spin reversal

$$\mathfrak{q}|\uparrow\rangle = 0, \quad \mathfrak{q}|\downarrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle - \zeta|\downarrow\rangle \otimes |\downarrow\rangle$$
$$\bar{\mathfrak{q}}|\downarrow\rangle = 0, \quad \bar{\mathfrak{q}}|\uparrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle - \zeta|\uparrow\rangle \otimes |\uparrow\rangle$$

and the gauge supercharge

$$\mathfrak{q}_\alpha|\psi\rangle = |\psi\rangle \otimes |\alpha\rangle + |\alpha\rangle \otimes |\psi\rangle$$

with $|\alpha\rangle \in V$. We consider the linear combination

$$\mathfrak{q}(y) = (1-y^2\zeta)\mathfrak{q}+y(y^2-\zeta)\bar{\mathfrak{q}}+\mathfrak{q}_\phi, \quad |\phi\rangle = y(y^2\zeta-1)|\uparrow\rangle+(\zeta-y^2)|\downarrow\rangle.$$

Supercharge for the open chain

$$\mathfrak{Q}(y) = \sum_{j=1}^{L}(-1)^j\mathfrak{q}_j, \quad \mathfrak{Q}(y)^2 = 0$$

XYZ Hamiltonian with open b.c.

Supersymmetry of the open chain

The anticommutator of the supercharge and its adjoint is

$$\frac{1}{\mathcal{N}}\left(\mathfrak{Q}(y)\mathfrak{Q}(y)^\dagger + \mathfrak{Q}(y)^\dagger\mathfrak{Q}(y)\right) = H_{XYZ} - E_0$$

where $\mathcal{N}$ is a positive normalisation constant and

$$J_x = 1+\zeta, \quad J_y = 1-\zeta, \quad J_z = \frac{1}{2}(\zeta^2-1),$$

$$\lambda_x = -\frac{J_x(y+y^*)}{2(1+|y|^2)}, \quad \lambda_y = -\frac{J_y(y-y^*)}{2i(1+|y^2|)}, \quad \lambda_z = \frac{J_z}{2}\left(\frac{1-|y|^2}{1+|y|^2}\right).$$

The Susy is present on all the space V^L.

We parametrise y by t : $y = \vartheta_1(t, p^2)/\vartheta_4(t, p^2)$.

For each $L \geqslant 1$, if $t \neq \pi/6$, then the (co)homology is trivial

$$\mathcal{H}^L = \mathcal{H}_L = 0.$$

If $t = \pi/6$, then

$$\mathcal{H}^L = \mathbb{C}[|v \cdots v\rangle], \quad \mathcal{H}_L = \mathbb{C}[|w \cdots w\rangle]$$

where $|v\rangle \in V^2$ and $|w\rangle \in V^2$ are known.

We do not present the proof of this stament.

- $\rightarrow$ Jacobi ϑ functions,
- $\rightarrow$ Ground states of the open XYZ spin chain

$$|\Psi_L\rangle = |v \cdots v\rangle + \mathfrak{Q}|\alpha\rangle = \mu|w \cdots w\rangle + \mathfrak{Q}^\dagger|\beta\rangle.$$

Transfer matrix

Double rows transfer matrix

$$\mathcal{T} = \mathrm{tr}_0 \left(K_0^+ R_{0L} \cdots R_{01} K_0^- R_{01} \cdots R_{0L} \right)$$

The R and $K^\pm$ matrix satisfies the Yang-Baxter equation and boundary YBE, respectively.
We take a specific (symmetric) solution for the K matrix

$$K = 1 + \frac{2y}{1+y^2}\frac{cd+ab}{ac+bd}\sigma^x + \frac{1-y^2}{1+y^2}\frac{b^2-d^2}{2ab+b^2+d^2}\sigma^z$$

that leads to the supersymmetric H_{XYZ}.

[Sklyanin '87, Inami & Konno '94]

Commutation relation between $\mathcal{T}$ and $\mathfrak{Q}$ – I.

There is an explicit mapping $A : V \to V \otimes V$, such that, defining $A_0^j : V_0 \otimes V^L \to V_0 \otimes V^{L+1}$ by

$$A_0^1 = A \otimes 1 \otimes \cdots \otimes 1, \quad A_0^j = \mathcal{S}^{j-1} A_0^1 \mathcal{S}^{1-j}, \quad j = 1, \ldots, L,$$

we have

Local commutation relation

$$R_{0i+1} R_{0i} (1 \otimes \mathfrak{q}_i(y)) + (a+b)(1 \otimes \mathfrak{q}_i(y)) R_{0i} = A_0^{i+1} R_{0i} + R_{0i+1} A_0^i$$

and a local commutation relation at the boundary,

$$(a+b) A_0^1 K_0 = R_{01} K_0 A_0^1$$

if $(a^2+ab)(b^2+ab) = (c^2+ab)(d^2+ab)$.

Commutation relation between $\mathcal{T}$ and $\mathfrak{Q}$ – II.

On V^L, $L \geqslant 1$

$$\mathcal{T}\mathfrak{Q}(y) = (a+b)^2\mathfrak{Q}(y)\mathcal{T}.$$

Proof: direct application of local commutation relations.

For the XXZ case $(\zeta = y = 0)$: arXiv:1709.00442 [Weston & Yang, '17]

For $t = \pi/6$, on can compute the action of the transfer matrix on Susy singlets $|\Psi_L\rangle$. We find

$$\mathcal{T}|\Psi_L\rangle = \Theta_L|\Psi_L\rangle, \quad \Theta_L = (a+b)^{2L}\mathrm{tr}(K^+K^-)$$

Proof: based on the structure of the representative and a reccurence on L : $\Theta_L = (a+b)^4\Theta_{L-2}$

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Conclusions

Conclusions

- Periodic b.c.: $(a+b)^{2n+1}$.
- Commutation relation between R and $\mathfrak{q}$:

  Integrability $\leftrightarrow$ supersymmetry

- Results for open b.c.

Generalisations:

- Inhomogeneous eigenvalue $\prod_{i=1}^{L}(a(u_i)+b(u_i))$?
- Explicit components of the eigenstates ?
- Computation of correlation functions ?

Thanks for your attention – OBRIGADO

Periodic b.c.: `arXiv`:1711.04397

Open b.c.: to be published ($2017 + \epsilon$, $\epsilon > 0$)