The six vertex model with various boundary conditions

Ivar Lyberg

International Institute of Physics

29 June 2018

Exactly Solvable Quantum Chains, IIP Natal 18-29 June 2018

Collaborators

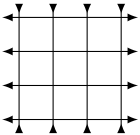
V. Korepin, Stony Brook
G. A. P. Ribeiro, São Carlos
J. Viti, Natal

- I. L., V. Korepin, and J. Viti, J. Stat. Mech. **2017** 053103. The density profile of the six vertex model with domain wall boundary conditions.
- I. L., V. Korepin, G. A. P. Ribeiro, and J. Viti, J. Math. Phys. **59**, 053301 (2018). Phase separation in the six-vertex model with a variety of boundary conditions.

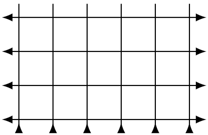
Boundary conditions

Domain Wall Boundary conditions V. Korepin, Commun. Math. Phys. **86**, 391 (1982)., Partial Domain Wall Boundary conditions O. Foda and M. Wheeler, J. High Energy Phys. **7**, 186 (2012). , Reflective End Boundary conditions O. Tsuchiya, J. Math. Phys. **39**, 5946 (1998)., Half Turn Boundary conditions P. Bleher and K. Liechty, Constr. Approximation **47**, 141 (2017). .

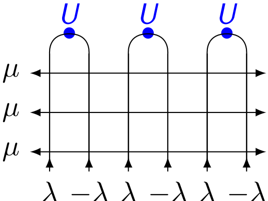
DWBC, $N = 4$



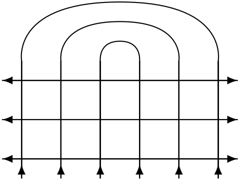
PDWBC; $M = 6$, $N = 4$



REBC, $N = 3$



HTBC, $N = 3$



Boundary conditions

1. Domain Wall Boundary Conditions. V. Korepin, Commun. Math. Phys. **86**, 391 (1982).

- The partition function can be written as a determinant. A. G. Izergin, Sov. Phys. Dokl. **32**, 878 (1987).
- The free energy is different from the free energy of the six vertex model with periodic or free boundary conditions. V. E. Korepin and P. Zinn-Justin, J. Phys. A **33**, 7053 (2000)

  Subleading corrections to the free energy. P. Bleher and V. Fokin, Commun. Math. Phys. **1** 223 (2006), P. Bleher and K. Liechty, Commun. Math. Phys. **286** 777 (2009).
- Severel phases coexist. O. F. Syljuåsen and M. B. Zvonarev Phys. Rev. E **70**, 016118 (2004); D. Allison and N. Reshetikhin, Ann. Inst. Fourier **56** (6), 1847 (2005).
- For a certain choise of parameters, the six vertex model with DWBC is equivalent to the domino tilings on the Aztec diamond. *Arctic Circle Theorem* W. Jockusch, J. Propp, and P. Shor, arXiv:math.CO/9801068.

Boundary conditions

2. Partial Domain Wall Boundary Conditions. O. Foda and M. Wheeler, J. High Energy Phys. **7**, 186 (2012).

- The model has three independent parameters. P. Bleher and K. Liechty, J. Math. Phys. **56** 023302 (2015).
- Separation of phases.

3. Reflective End Boundary Conditions. O. Tsuchiya, J. Math. Phys. **39**, 5946 (1998).

- The model has three independent parameters.
- The free energy is different from the free energy of the model with periodic boundary conditions. G. A. P. Ribeiro and V. E. Korepin, J. Phys. A **48**, 045205 (2015).

4. Half Turn Boundary Conditions. P. Bleher and K. Liechty, Constr. Approximation **47**, 141 (2017).

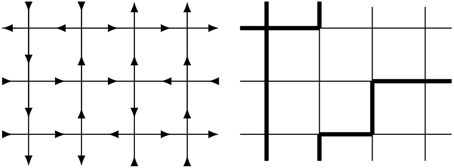
- The free energy is the same as with DWBC Bleher and Liechty (2017).

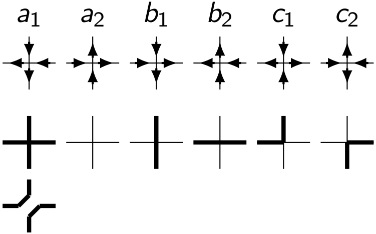
Plan

1. The six vertex model
(i) Introduction
(ii) Domain wall boundary conditions and Monte Carlo algorithm
(iii) Partial domain wall boundary conditions
(iv) Reflective end boundary conditions
(v) Half turn boundary conditions
2. The nineteen vertex model with domain wall boundary conditions

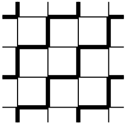
Introduction - Notation

$M = 4,\ N = 3$





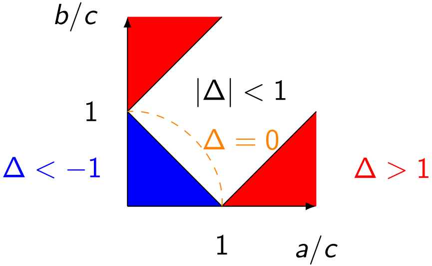
Antiferromagnetic phase



- Six Boltzmann weights $w(a_1)$, etc.
- The restriction $w(a_1) = w(a_2) = a$, $w(b_1) = w(b_2) = b$, $w(c_1) = w(c_2) = c$ is used. There are thus two independent parameters; a/c and b/c.

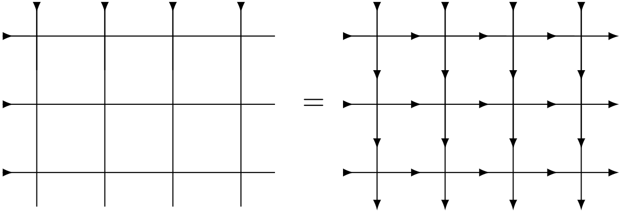
Introduction - Periodic and Free Boundary Conditions

- The free energy has been exactly calculated for periodic boundary conditions [Lieb (1967)] and for free boundary conditions [Brascamp, Kunz and Wu (1973)].
- The free energy is the same for periodic and free boundary conditions.
- The phase diagram is determined by the parameter $\Delta := (a^2 + b^2 - c^2)/2ab$.



Introduction - Fixed Boundary Conditions

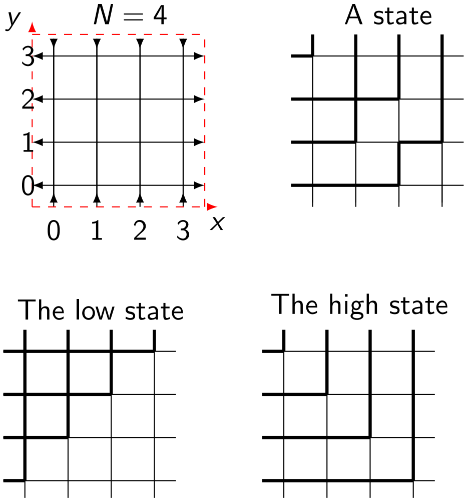
- The free energy depends on the boundary conditions, even in the thermodynamic limit.



- There may be separation of phases; several phases may coexist.
- There are exact solutions for three nontrivial cases: Domain Wall Boundary Conditions, Reflective End Boundary Conditions and Half Turn Boundary Conditions.

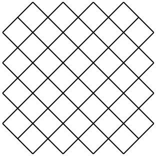
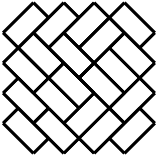
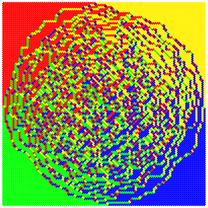
Domain Wall Boundary Conditions - Definition

- DWBC are defined on a square lattice of size $N \times N$
- Each state is a set of N curves which flow from the upper boundary to the left boundary



DWBC - Arctic Circle

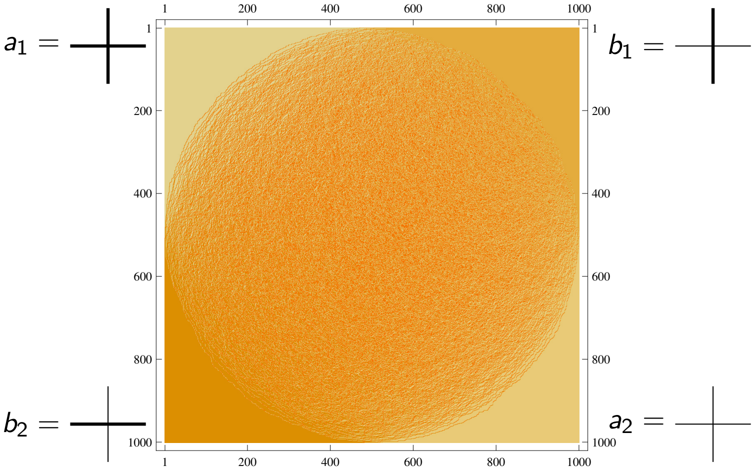
- The six vertex model with DWBC, $\Delta = 0$ and $a = b$ is equivalent to the domino tilings on the Aztec diamond. P. Ferrari and H. Spohn, J. Phys. A **39** 10297 (2006).

- Arctic Circle Theorem: The curve approaches a circle.
- Shape fluctuations of the arctic curve are of order $N^{1/3}$. K. Johansson, Commun. Math. Phys. **209** 437 (2000).
- There are also exact results on the arctic curve for $\Delta \neq 0$. F. Colomo and A. Pronko, J. Stat. Phys. **138** 662 (2010) and F. Colomo, A. Pronko and P. Zinn-Justin, J. Stat. Mech. L03002 (2010).

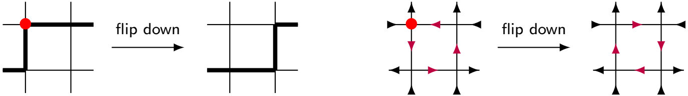
DWBC - Arctic Circle

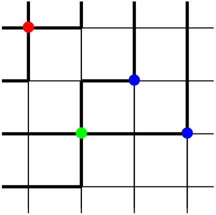
- The Arctic Circle Theorem can be illustarated by Monte Carlo.
- As N increases, the thermalization time increases rapidly.



DWBC - Monte Carlo

- We begin with any allowed state. As the program runs, it will then create a new allowed state at every step.
- A reasonable size is $N = 500$.
- Thermalization is slow when $\Delta << -1$.
- We use an algorithm first introduced by Allison and Reshetikhin. It uses perturbation of the curves. There are two Monte Carlo moves; flip up and flip down.
- Not every vertex will be flippable.





Flippable down

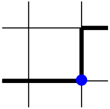
Flippable up

Flippable up and down

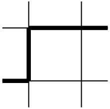
DWBC - Monte Carlo

- We consider a Markov process where p_{ab} is the probability of a transition from the state a to the state b. The matrix (p_{ab}) must satify the total probability condition $\sum_b p_{ab} = 1$.
- If q is a probability measure on the set of states, and (p_{ab}) satisfies the detailed balance condition $q_a p_{ab} = q_b p_{ba}$, then the Markov process will converge to q.
- We search randomly among all N^2 vertices until we find a vertex $v = (x, y)$ which is flippable.
- If we find a vertex flippable up (only), then we flip up with probability $P_{\text{up flip}}$. If the flip happens, the state S is replaced by a new state S'.

$$S = \begin{pmatrix} a_2 & c_2 \\ b_2 & c_1 \end{pmatrix} =$$



$$\xrightarrow{\text{flip up}}$$

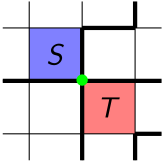


$$= \begin{pmatrix} c_2 & b_2 \\ c_1 & a_2 \end{pmatrix} = S'$$

$$P_{\text{up flip}} = \frac{W(S')}{R} = \frac{w(a_2)w(b_2)w(c_1)w(c_2)}{R}.$$

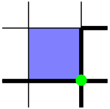
DWBC - Monte Carlo

- If we find a vertex flippable both up and down, a flip will happen with probability P_{flip}. The flip will then be up with probabilty $P_{\text{up flip} \mid \text{flip}}$.

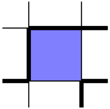


$$P_{\text{flip}} = \frac{W(S')+W(T')}{R}$$

$$S = \begin{pmatrix} a_2 & c_2 \\ b_2 & a_1 \end{pmatrix} =$$



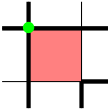
$$\xrightarrow{\text{flip up}}$$



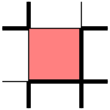
$$= \begin{pmatrix} c_2 & b_2 \\ c_1 & c_2 \end{pmatrix} = S'$$

$$P_{\text{up flip} \mid \text{flip}} = \frac{W(S')}{W(S')+W(T')} = \frac{w(b_2)w(c_1)w(c_2)^2}{W(S')+W(T')}.$$

$$T = \begin{pmatrix} a_1 & b_2 \\ b_1 & c_2 \end{pmatrix} =$$



$$\xrightarrow{\text{flip down}}$$

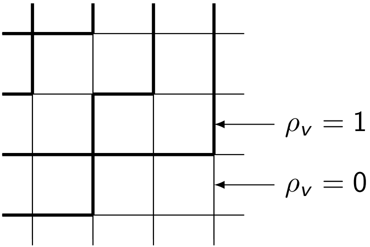


$$= \begin{pmatrix} c_1 & c_2 \\ c_2 & a_1 \end{pmatrix} = T'$$

$$P_{\text{down flip} \mid \text{flip}} = \frac{W(S')}{W(S')+W(T')} = \frac{w(a_1)w(c_1)w(c_2)^2}{W(S')+W(T')}.$$

DWBC - Density profiles

- One possible measure of the model is the density on, for example, vertical edges; ρ_v.



- When $\Delta = 0$, $\langle \rho_v \rangle$ is exactly known. N. Allegra, J. Dubail, J.-M. Stéphan and J. Viti, J. Stat. Mech. 053108 (2016). O. F. Syljuåsen and M. B. Zvonarev (2004).
- The function $\langle \rho_v(x, y) \rangle$ has oscillations which disappear in the limit $N \to \infty$. These oscillations become more pronounced as Δ decreases.

DWBC - Density profiles. $\Delta = 0$.

N. Allegra, J. Dubail, J.-M. Stéphan and J. Viti, J. Stat. Mech. 053108 (2016).

$$\langle \rho_v(x,y) \rangle = \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi} e^{i(k_2-k_1)x + y[\epsilon(\kappa(k_1)) - \epsilon(\kappa(k_2))]}$$

$$e^{iN[\tilde{\epsilon}(\kappa(k_2)) - \tilde{\epsilon}(\kappa(k_1))]} \sqrt{1 - i\frac{b}{a}\sin\kappa(k_1)} \sqrt{1 - i\frac{b}{a}\sin\kappa(k_2)}$$

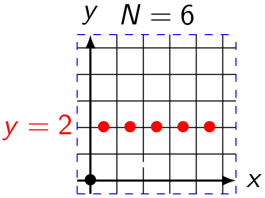
$$\sqrt{\kappa'(k_1)} \sqrt{\kappa'(k_2)} \frac{1}{2i \sin\frac{\kappa(k_1)-\kappa(k_2)}{2}}$$

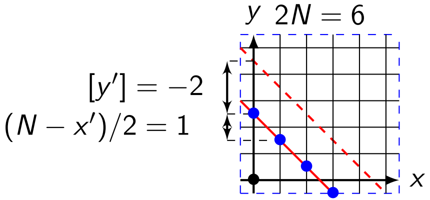
where

$$\tan\kappa(k) = \frac{a}{c}\tan k,\ \epsilon(\kappa) = -\frac{1}{2}\log\frac{c + b\cos\kappa}{c - b\cos\kappa},\ \tilde{\epsilon}(\kappa) = \frac{i}{2}\log\frac{a + ib\sin\kappa}{a - ib\sin\kappa}$$

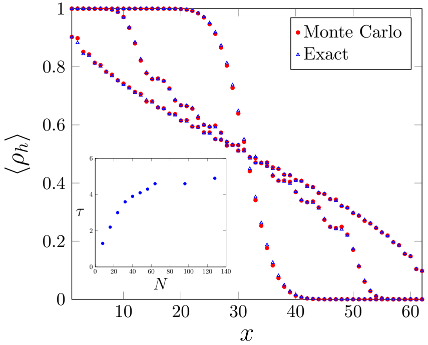
- In the limit $N \to \infty$ (x/N, y/N constant) the double integral becomes an elementary function.
- The integral is difficult to evaluate accurately for finite N.

DWBC - Density profiles. $\Delta = 0$, $a = b$.

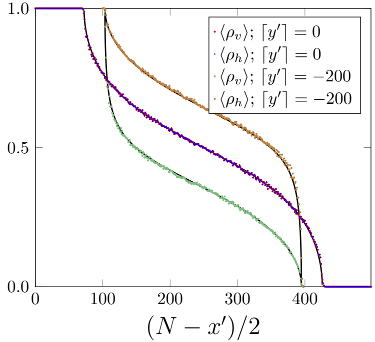




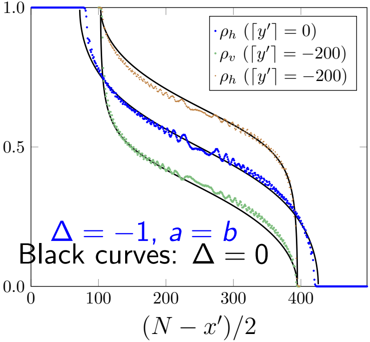
$N = 63$; $y = 0,\ 8,\ 30.$

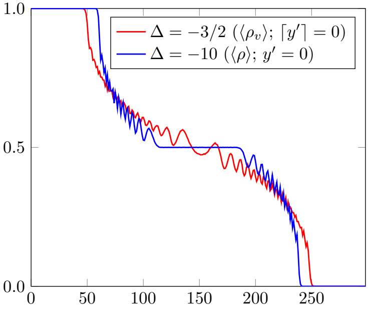


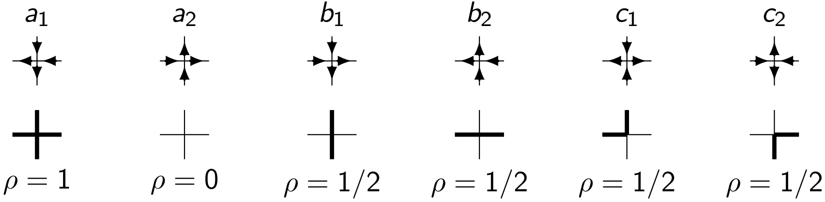
$N = 500.$



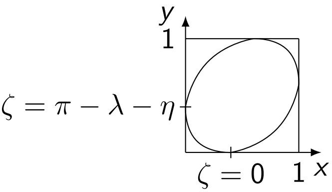
DWBC - Density profiles. $\Delta \leq -1$.







DWBC - Arctic curves. $|\Delta| < 1$. F. Colomo and A. Pronko (2010)

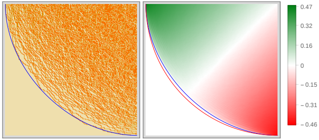


$$a = \sin(\lambda + \eta), \ \ b = \sin(\lambda - \eta)$$
$$c = \sin 2\eta, \ \ \Delta = \cos 2\eta$$
$$x = X(\zeta), \ \ y = Y(\zeta)$$
$$0 \le \zeta \le \pi - \lambda - \eta$$
$$X(\zeta) = Y(\pi - \lambda - \eta - \zeta)$$

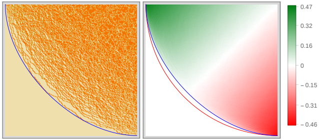
$$Y(\zeta) = \frac{\sin^2\zeta \sin^2(\zeta + 2\eta)\sin(\zeta + \lambda - \eta)\sin(\zeta + \lambda + \eta)}{\sin 2\eta \sin(\lambda - \eta)[\sin(\zeta + \lambda + \eta)\sin\zeta + \sin(\zeta + \lambda - \eta)\sin(\zeta + 2\eta)]}$$
$$\times \Big[\frac{\sin(\lambda - \eta)\sin(\lambda + \eta)}{\sin^2\zeta \sin(\zeta + \lambda - \eta)\sin(\zeta + \lambda + \eta)}$$
$$+ \frac{\sin(2\zeta + 2\lambda)}{\sin(\zeta + \lambda - \eta)\sin(\zeta + \lambda + \eta)} \frac{\alpha \sin\alpha(\lambda - \eta)}{\sin\alpha\zeta \sin\alpha(\zeta + \lambda - \eta)}$$
$$- \frac{\alpha^2 \sin\alpha(2\zeta + \lambda - \eta)\sin\alpha(\lambda - \eta)}{\sin^2\alpha\zeta \sin^2\alpha(\zeta + \lambda - \eta)}\Big]; \qquad \alpha = \frac{\pi}{\pi - 2\eta}$$

DWBC - Arctic curves. $a = b$.

$\Delta = -1/2$



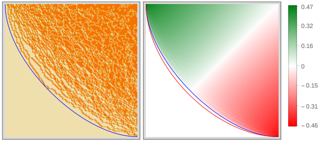
$\Delta = -1$



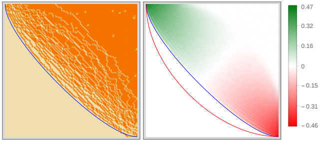
Arctic curve

$\Delta = 0$

$\Delta = -2$



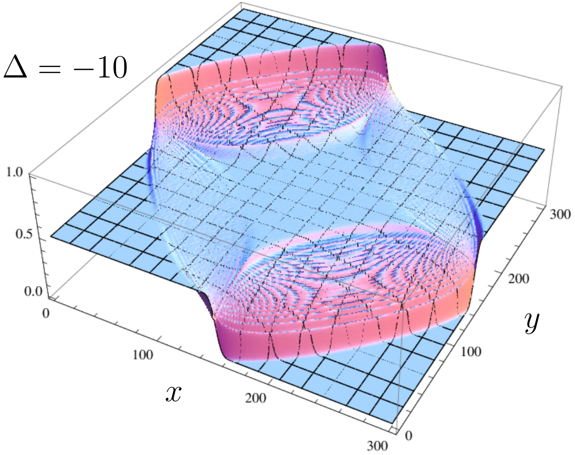
$\Delta = -10$



Arctic curve

$\Delta = -1$

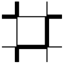
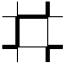
DWBC - Antiferromagnetic regime ($\Delta < -1$)

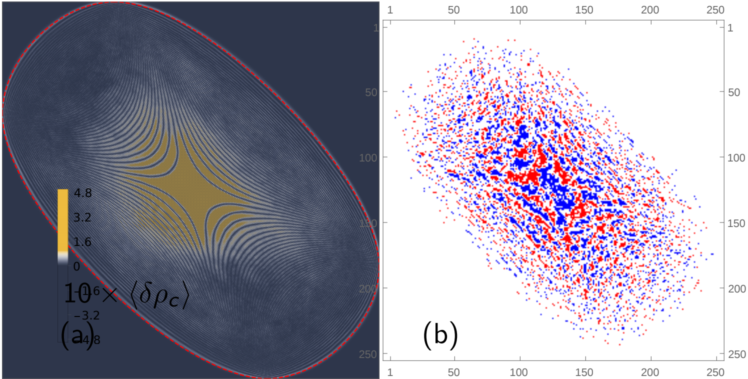


- Ferromagnetic, disordered and antiferromagnetic phases coexist.
- The inner curve has no known equation.
- "Saddle points" in the disordered region.

DWBC - Antiferromagnetic regime ($\Delta < -1$)

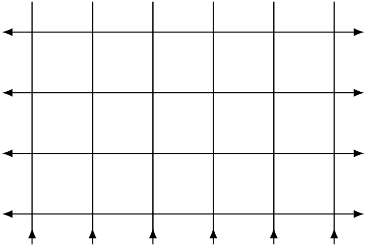
- $\Delta = -2$, $b = 2a$.
- (a) The measured arctic curve agrees with the exact arctic curve. Quantity measured is $\langle \delta\rho_c \rangle := \langle \rho_{c_1} \rangle - \langle \rho_{c_2} \rangle$.
- (b) Map of the two opposite a.f.m. phases in a typical state.

 $= \begin{pmatrix} c_1 & c_2 \\ c_2 & c_1 \end{pmatrix}$  $= \begin{pmatrix} c_2 & c_1 \\ c_1 & c_2 \end{pmatrix}$

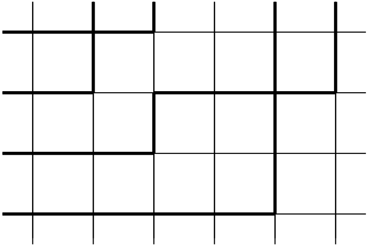


Partial Domain Wall Boundary Conditions

$M = 6,\ N = 4$



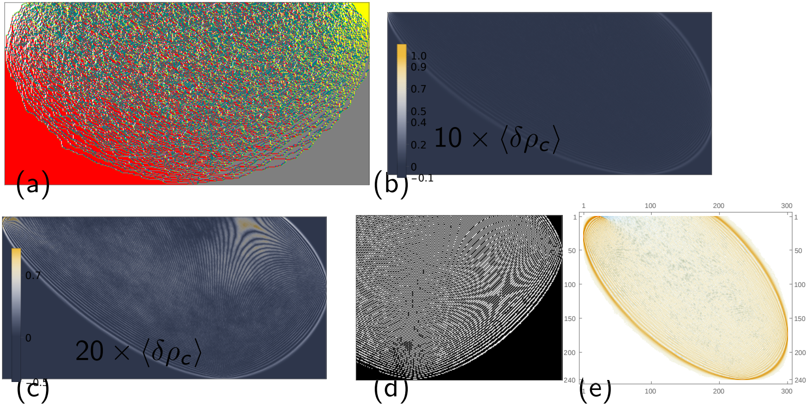
A state



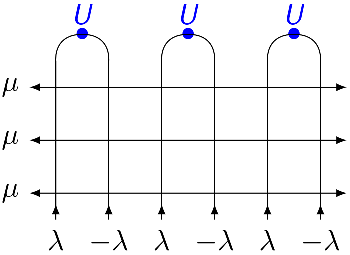
- Free upper boundary, $M \geq N$.
- A state consists of N curves beginning on the upper boundary and ending on the left boundary.
- There are no exact results.
- There are five weights; $w(a_1)$, $w(a_2)$, $w(b_1)$, $w(b_2)$ and $w(c_1) = w(c_2) = c$. There are three independent parameters; we have the restriction $c = 1$ and $w(a_1)/w(a_2) = w(b_1)/w(b_2)$.

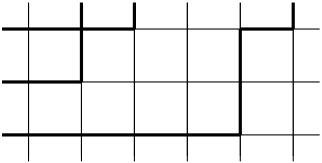
Partial DWBC - Examples ($M/N = 2$, $M/N = 5/4$)

- (a) $\Delta = 0$, $a = b$. Four frozen corners.
- (b) $\Delta = 0$, $b = 2a$.
- (c), (d) $\Delta = -2$, $b = 2a$. Two antiferromagnetic regions.
- (e) $M/N = 5/4$, $\Delta = 0$, $b = 2a$.



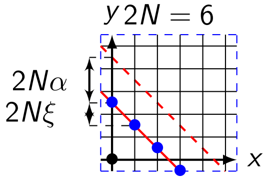
Reflective End Boundary Conditions



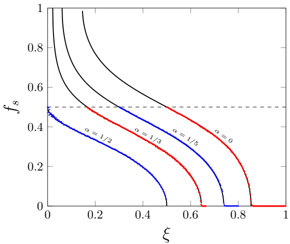


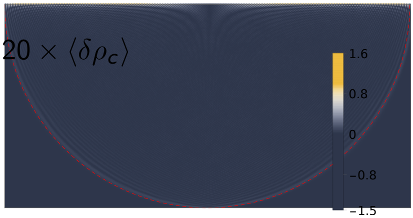
- $M = 2N$
- Three spectral parameters γ, μ, λ. Here $\mu = 0$.
- In the disordered regime $(-1 < \Delta < 1)$ $a(\lambda) = \sin(\gamma - \lambda)$, $b(\lambda) = \sin(\gamma + \lambda)$ and $c(\lambda) = \sin 2\gamma$.
- On even columns, $w(a_1) = w(a_2) = a(\lambda)$, $w(b_1) = w(b_2) = b(\lambda)$ and $w(c_1) = w(c_2) = c(\lambda)$. On odd columns $w(a_1) = w(a_2) = b(\lambda)$, $w(b_1) = w(b_2) = a(\lambda)$.

Reflective End BC. The case $\Delta = 0$, $a = b$.



- (a) The density profile on vertical edges is the same as for DWBC.
- (b) In particular, the arctic curve is a semicircle.

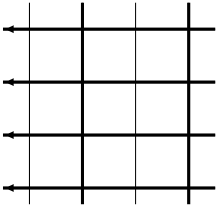




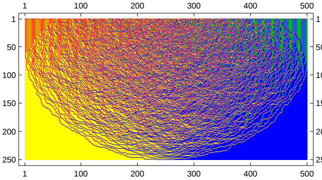
Reflective End BC. The case $\Delta = 0$, $b = 2a$.

- When $b > a$, there are two new frozen corners at the upper boundary. The arctic curve is not known
- Figure (a) shows the structure of the upper left corner.

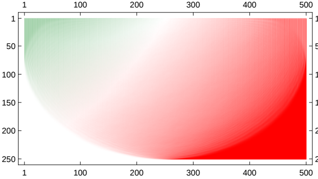
(a)



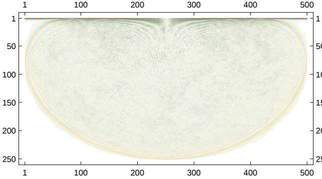
(b)



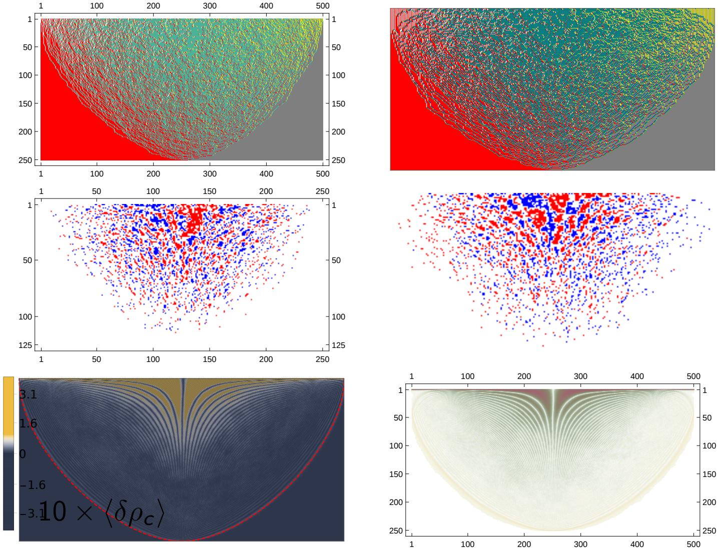
(c)



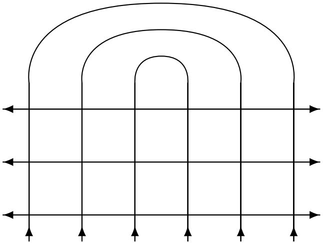
(d)

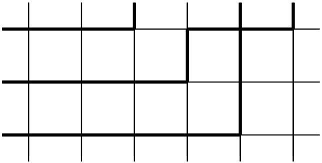


Reflective End BC. The cases $\Delta = -2$; $b = a$, $b = 2a$.

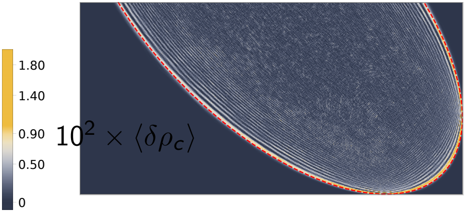


Half Turn Boundary Conditions





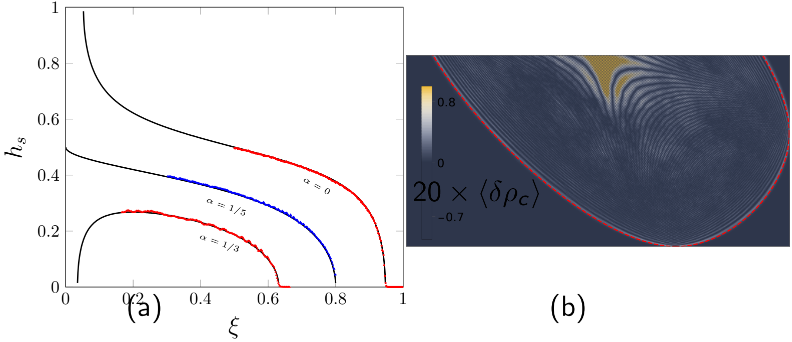
- $M = 2N$.
- Numerical results indicate that the arctic curves are the same as for DWBC.



$N = 250$

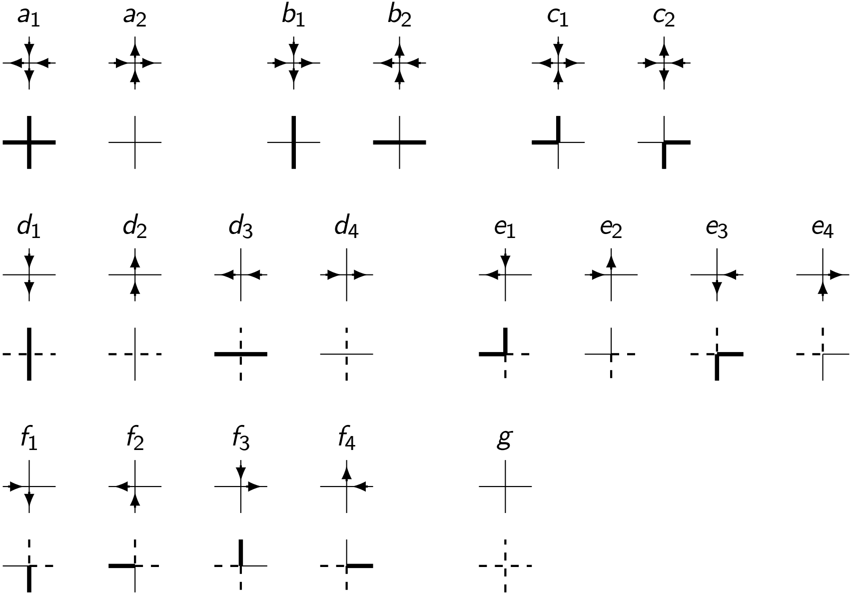
$\Delta = 0,\ b = 2a$

Half Turn Boundary Conditions

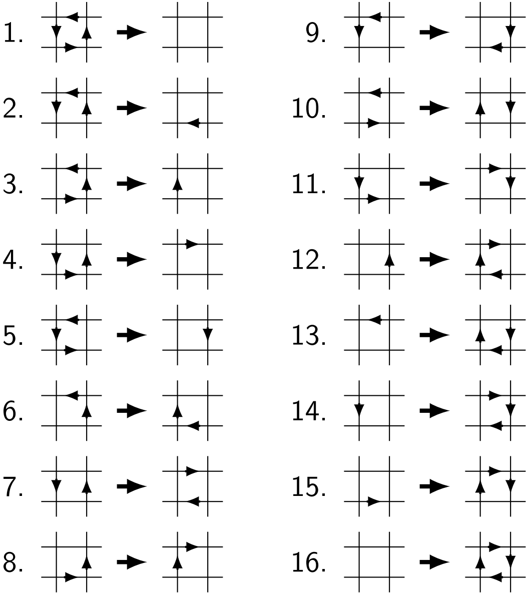


- (a) For $N = 500$, $\Delta = 0$ and $b = 2a$ the density profile on vertical edges agrees with the asymptotic formula for DWBC with $\Delta = 0$ and $b = 2a$.
- (b) For $N = 500$, $\Delta = -2$ and $b = 2a$ the arctic curve agrees with the arctic curve for DWBC with $\Delta = -2$ and $b = 2a$.

19 vertex model with DWBC - The 19 vertices



19 vertex model with DWBC - The 32 flips



19 vertex model with DWBC - Previous work

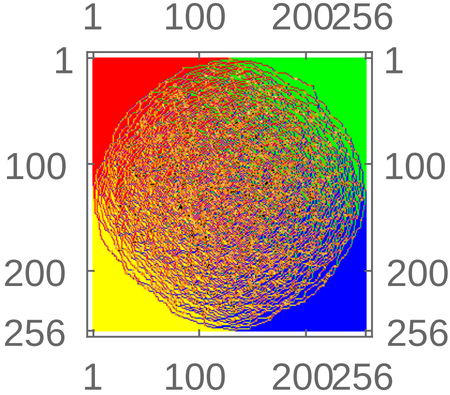
- In the integrable case, the weights are parametrized by two variables. A. Klümper, M. T. Batchelor and P. A. Pearce, J. Phys. A **24** 3111 (1991).

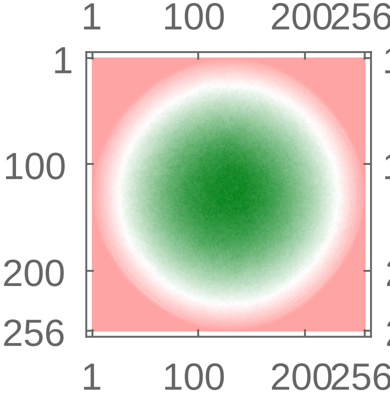
$$
\begin{aligned}
w_a &= \sinh(\lambda - u)\sinh(2\lambda - u) \\
w_b &= \sinh u \sinh(\lambda + u) \\
w_c &= \sinh\lambda \sinh 2\lambda \\
w_d &= \sinh u \sinh(\lambda - u) \\
w_e &= \sinh 2\lambda \sinh(\lambda - u) \\
w_f &= \sinh 2\lambda \sinh u \\
w_g &= \sinh\lambda \sinh 2\lambda - \sinh u \sinh(\lambda - u)
\end{aligned}
$$

- The model is critical when $\lambda = -i\gamma$, $0 \leq \gamma < \pi$.
- The model with these boundary conditions has recently been studied by Monte Carlo K. Eloranta, arXiv: 1710.03609

19 vertex model with DWBC - Monte Carlo

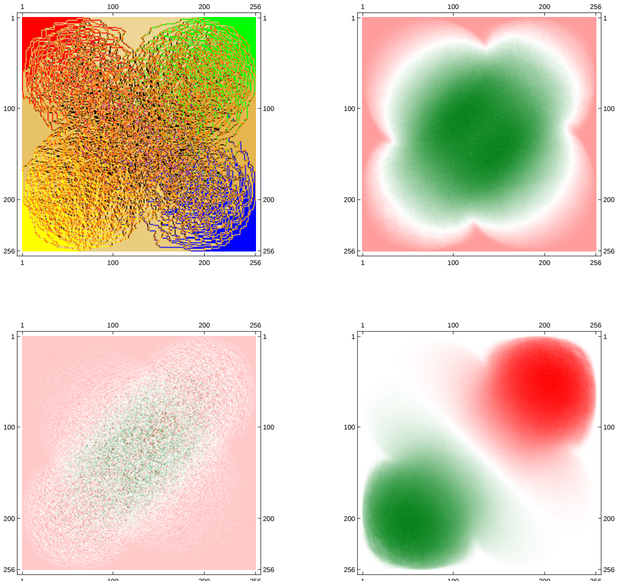
- $\lambda = -i\pi/3$, $u = -i\pi/6$
- Frozen corners, disorder in center.





19 vertex model with DWBC - Monte Carlo

- $w_a = w_b = w_e = 1$, $w_c = w_g = 2.5$, $w_d = w_f = 2$.



Conclusion and prospects

- The six vertex model with Partial DWBC displays phase separation and there are frozen corners as in the six vertex model with DWBC.
- The arctic curves of the six vertex model with Partial DWBC are unknown.
- The six vertex model with REBC displays phase separation and there are frozen corners. When $a = b$ the arctic curve is the same as for DWBC. When $a \neq b$ the arctic curve is different from the arctic curve for DWBC, and there are two additional frozen corners.
- The arctic curves of the six vertex model with REBC are unknown.
- The six vertex model with HTBC is the same as the six vertex model with DWBC in the thermodynamic limit.
- It should be possible to study the 19-vertex model with a similar algorithm.