λ-models as Chern-Simons theories

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Presented at EXACTLY SOLVABLE QUANTUM CHAINS, Natal, RN, Brasil

Based on e-Print: arXiv:1701.04138 and work in preparation.

June 28 2018

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Green-Schwarz σ-model on $AdS_5 \times_\sigma S^5$

Consider the GS action on the worldsheet $\Sigma = S^1 \times \mathbb{R}$

$$S_\sigma = -4g \int_\Sigma d^2\sigma \, \langle J_+ \theta J_- \rangle \,, \quad \theta = P^{(2)} - \frac{1}{2}(P^{(1)} - P^{(3)}),$$

where $J_\pm = f^{-1}\partial_\pm f \in \mathfrak{psu}(2,2|4)$ and $f \in PSU(2,2|4)$.
The Euler-Lagrange EOM can be put in Lax form with Lax pair

$$\mathscr{L}_\pm(z) = J_\pm^{(0)} + zJ_\pm^{(1)} + z^{\pm 2} J_\pm^{(2)} + z^{-1} J_\pm^{(3)}.$$

A convenient form to start the Hamiltonian analysis is to write instead

$$S_\sigma = -4g \int_\Sigma d^2\sigma \, \langle A_+ \theta A_- + \nu F_{+-} \rangle \,,$$

where ν is a Lagrange multiplier enforcing the condition $F_{+-} = 0$. This last expression can be interpreted as well as a non-Abelian version of the Buscher approach to T-duality.

The λ-model in $AdS_5 \times_\lambda S^5$

A λ-model is an integrable deformation of a σ-model: arXiv:1409.1538

$$S_\lambda = S^{WZW}_{F/F}(\mathcal{F}, A_\pm) - \frac{k}{\pi}\int_\Sigma d^2\sigma \left\langle A_+(\Omega - 1)A_- \right\rangle, \quad k \in \mathbb{Z},$$

where $\mathcal{F} \in PSU(2,2|4)$, $A_\pm \in \mathfrak{psu}(2,2|4)$ and

$$\Omega(\lambda) = P^{(0)} + \lambda P^{(1)} + \lambda^{-2}P^{(2)} + \lambda^{-1}P^{(3)}, \quad \lambda^{-2} = 1 + \frac{4\pi g}{k}.$$

For $\lambda \to 1$ with $k \to \infty$, $g =$ fixed and $\mathcal{F} = 1 + \frac{4\pi g}{k}\nu$, the action reduces to the "Non-Abelian Buscher's" form

$$S_\sigma = -4g\int d^2\sigma \left\langle A_+\theta A_- + \nu F_{+-} \right\rangle.$$

For $\lambda \to 0$ with $g \to \infty$, $k =$fixed, we get a current-current perturbation of a gauged WZW model, intimately related to the Pohlmeyer reduction.

Lax pair and key relations

Euler-Lagrange EOM can be put in Lax form

$$\mathscr{L}_\pm(z) = I_\pm^{(0)} + zI_\pm^{(1)} + z^{\pm 2} I_\pm^{(2)} + z^{-1} I_\pm^{(3)},$$

in terms of the $\mathfrak{psu}(2,2|4)$ "dual" currents

$$\begin{aligned} I_+ &= \Omega^T(\lambda^{1/2})[\Omega^T(\lambda) - Ad_{\mathcal{F}^{-1}}]^{-1}\mathcal{F}^{-1}\partial_+\mathcal{F}, \\ I_- &= -\Omega^{-1}(\lambda^{-1/2})[\Omega(\lambda) - Ad_{\mathcal{F}}]^{-1}\partial_-\mathcal{F}\mathcal{F}^{-1}. \end{aligned}$$

The $A_\pm$ EOM are equivalent to

$$\mathscr{J}_+ = \frac{k}{2\pi}(\Omega^T(\lambda)A_+ - A_-), \quad \mathscr{J}_- = -\frac{k}{2\pi}(A_+ - \Omega(\lambda)A_-),$$

where

$$\begin{aligned} \mathscr{J}_+ &= \frac{k}{2\pi}(\mathcal{F}^{-1}\partial_+\mathcal{F} + \mathcal{F}^{-1}A_+\mathcal{F} - A_-), \\ \mathscr{J}_- &= -\frac{k}{2\pi}(\partial_-\mathcal{F}\mathcal{F}^{-1} - \mathcal{F}A_-\mathcal{F}^{-1} + A_+), \end{aligned}$$

are two mutually commuting Kac-Moody currents

As a consequence of this we have that ($z_\pm = \lambda^{\pm 1/2}$)

$$\mathscr{L}_\sigma(z_\mp) = \pm\frac{2\pi}{k}\mathscr{J}_\pm \Longrightarrow m(z_\mp) = P\exp[\mp\frac{2\pi}{k}\int_{S^1} d\sigma\, \mathscr{J}_\pm].$$

Now, on-shell, and in terms of the wave function

$$(\partial_\mu + \mathscr{L}_\mu(z))\Psi(z) = 0,$$

we can expresse the Lagrangian fields in the form

$$\mathcal{F} = \Psi(z_+)\Psi(z_-)^{-1}, \qquad A_\pm = -\partial_\pm\Psi(z_\pm)\Psi(z_\pm)^{-1},$$
$$\Omega^T(\lambda)A_+ = -\partial_+\Psi(z_-)\Psi(z_-)^{-1}, \quad \Omega(\lambda)A_- = -\partial_-\Psi(z_+)\Psi(z_+)^{-1}.$$

From this follows that

$$S_{\text{on-shell}} = S_{WZW}(\Psi(z_+)) - S_{WZW}(\Psi(z_-))$$

signaling a phase space decomposition at the points $z_\pm$. The key relations for $m(z_\pm)$ and $\mathcal{F}$ suggest a connection with a Chern-Simons theory.

General properties of λ-models...

- Introduced by Sfetsos in arXiv:1312.4560 for the PCM.
- Preserves integrability but breaks the global F left action Noether symmetry of the σ-model that reemerges as a Poisson-Lie group signaling a quantum group $\mathcal{U}_q(\mathfrak{f})$ symmetry. arXiv:1506.06601
- Works as a regularization of the σ-model spectrum which is truncated by the WZW level k. arXiv:1704.05437
- Implement the Faddeev-Reshetikhin ultra-localization mechanism directly in the action functional. arXiv:1506.06601
- The theory when $\lambda \to 0$ in the Green-Schwarz case is naturally connected with the Pohlmeyer reduction. arXiv:1407.2840
- In the Green-Schwarz case the action possesses a deformed version of kappa symmetry. arXiv:1409.1538

- The gauge fixed theory is a symplectic deformation of the σ-model with a dispersion relation that breaks 2D Lorentz symmetry in a controlled way. arXiv:1704.05437
- The beta function exact in λ but to one-loop in $1/k$ for the Green-Schwarz $AdS_5 \times_\lambda S^5$ vanishes. arXiv:1507.05420
- The beta function exact in λ but to one-loop in $1/k$ for the $AdS_2 \times_\lambda S^2$ Hybrid formalism vanishes. arXiv:1609.05330
- For $\lambda \to 0$ in the Hybrid case above, the light-cone current components along coset directions does not mix. arXiv:1609.05330
- In the λ-model for the Green-Schwarz $AdS_2 \times S^2 \times T^6$, the background fields solve 10D SUGRA eom. (Weyl invariant at quantum level hence a string background). arXiv:1601.08192. arXiv: 1606.00394 for n=3 and arXiv: 1608.03570 for n=5.

Integrability and Hamiltonian analysis

Main goal

To exploit the λ-model/Chern-Simons theory link in order to bypass the problem of the non-ultralocality of the deformed superstring.

The $\mathscr{L}_\sigma(z)$ does not obey the Maillet algebra. *Reason: We are dealing with a constrained integrable field theory.*
This is fixed by constructing a Lax pair extension $\overline{\mathscr{L}}_\mu(z)$ outside the constrained surface such that:

- The extended connection $\overline{\mathscr{L}}_\mu(z)$ is strongly flat, i.e. it is flat on the whole phase space.
- The extended monodromy matrix $\overline{m}(z)$ is first class, i.e. it preserves the constrained surface where the lambda model motion takes place.

This allows to identify sistematically the CS theory and find an ultralocal Poisson bracket for Wilson loops at the boundary of the disc at the expense of introducing two new first class constraints.

Dirac procedure

Poisson brackets on lambda model phase space:

$$\begin{aligned}
\{\mathscr{J}_{\pm 1}(\sigma), \mathscr{J}_{\pm 2}(\sigma')\} &= -[C_{12}, \mathscr{J}_{\pm 2}(\sigma')]\delta_{\sigma\sigma'} \mp \frac{k}{2\pi} C_{12}\delta'_{\sigma\sigma'}, \\
\{P_{\pm 1}(\sigma), A_{\mp 2}(\sigma')\} &= \frac{1}{2} C_{12}\delta_{\sigma\sigma'}.
\end{aligned}$$

Canonical Hamiltonian:

$$\begin{aligned}
H_C = -\frac{k}{\pi}\Big\langle \left(\frac{\pi}{k}\right)^2 \left(\mathscr{J}_+^2 + \mathscr{J}_-^2\right) + \frac{2\pi}{k}\left(A_+ \mathscr{J}_- + A_- \mathscr{J}_+\right) \\
+ \frac{1}{2}\left(A_+^2 + A_-^2\right) - A_+ \Omega A_- \Big\rangle.
\end{aligned}$$

Primary constraints:

$$P_\pm \approx 0.$$

Secondary constraints:

$$C_+ = \mathscr{J}_+ - \frac{k}{2\pi}(\Omega^T A_+ - A_-) \approx 0, \quad C_- = \mathscr{J}_- + \frac{k}{2\pi}(A_+ - \Omega A_-) \approx 0.$$

Equivalent to the $A_\pm$ EOM.

Extended Hamiltonian:

$$H_E = H_T - 2\left\langle u_+ P_- + u_- P_+ + \mu_+ C_- + \mu_- C_+ \right\rangle.$$

Running again, stability of the constraints under the flow of H_E fixes several Lagrange multipliers and produce no tertiary constraints.
In particular (kappa symmetry)

$$\mu_-^{(1)} = -A_-^{(1)} + [A_+^{(2)}, \kappa^{(1)}]_+, \quad \mu_+^{(3)} = -A_+^{(3)} + [A_-^{(2)}, \kappa^{(3)}]_+.$$

First class primary constraints

$$P_+^{(0)} + P_-^{(0)} \approx 0, \; z_+ P_+^{(1)} + z_- P_-^{(1)} \approx 0, \;\; z_- P_+^{(3)} + z_+ P_-^{(3)} \approx 0$$

are gauge fixed by

$$A_-^{(0)} \approx 0, \;\; A_-^{(1)} \approx 0, \;\; A_+^{(3)} \approx 0.$$

Second class pairs:

$$\begin{aligned} P_+^{(0)} - P_-^{(0)} &\approx 0, \quad C_-^{(0)} \approx 0, \\ z_+ P_+^{(1)} - z_- P_-^{(1)} &\approx 0, \quad C_-^{(3)} \approx 0, \\ z_- P_+^{(3)} - z_+ P_-^{(3)} &\approx 0, \quad C_-^{(1)} \approx 0. \end{aligned}$$

At the end of the day: The Kac- Moody algebra is protected, the fields $P_\pm$ are eliminated, the constrains can be imposed strongly and we have that

$$\begin{aligned} I_1^{(0)} &= -\frac{2\pi}{k}\mathscr{J}_-^{(0)}, \quad I_1^{(1)} = -\frac{2\pi}{k}z_-\mathscr{J}_-^{(1)}, \quad I_1^{(3)} = -\frac{2\pi}{k}z_+\mathscr{J}_-^{(3)}, \\ I_+^{(2)} &= \alpha(z_-^2\mathscr{J}_+^{(2)} + z_+^2\mathscr{J}_-^{(2)}), \quad I_-^{(2)} = \alpha(z_+^2\mathscr{J}_+^{(2)} + z_-^2\mathscr{J}_-^{(2)}), \end{aligned}$$

where $\alpha = -(2\pi/k)(z_+^4 - z_-^4)^{-1}$. The only remaining constraints are

$$\begin{aligned} \varphi^{(0)} &= C_+^{(0)} = \mathscr{J}_+^{(0)} + \mathscr{J}_-^{(0)}, \\ \varphi^{(1)} &= C_+^{(1)} = \mathscr{J}_+^{(1)} + z_-^2\mathscr{J}_-^{(1)}, \\ \varphi^{(3)} &= C_+^{(3)} = \mathscr{J}_+^{(3)} + z_+^2\mathscr{J}_-^{(3)} \end{aligned}$$

and the phase space is entirely parameterized by the Kac-Moody currents. In this partial gauge, the Lax pair takes the form

$$\mathscr{L}_+(z) = I_1^{(0)} + zI_1^{(1)} + z^2 I_+^{(2)}, \quad \mathscr{L}_-(z) = -z^{-1}I_1^{(3)} + z^{-2}I_-^{(2)}$$

as expansions around $z = 0$ and $z = \infty$.

In this conformal gauge approach the Virasoro constraints are imposed by hand. The first class (shifted) Virasoro constraints are

$$T'_{++} = T_{++} - z_+ \langle I_1^{(1)} \varphi^{(3)} \rangle, \quad T'_{--} = T_{--} + z_+ \langle I_1^{(3)} \varphi^{(1)} \rangle,$$

where

$$\begin{aligned} T_{++} &= -\frac{1}{2\alpha} \langle I_+^{(2)} I_+^{(2)} \rangle - \langle \frac{\pi}{k} (\varphi^{(0)} \varphi^{(0)} + 2\varphi^{(1)} \varphi^{(3)}) - (z_+ - z_-) I_1^{(1)} \varphi^{(3)} \rangle, \\ T_{--} &= -\frac{1}{2\alpha} \langle I_-^{(2)} I_-^{(2)} \rangle \end{aligned}$$

comes from the variation with respect to the 2D metric.
The momentum generator (requires a further shifting)

$$\overline{P} = T'_{++} - T'_{--} - \langle I_1^{(0)} \varphi^{(0)} \rangle$$

satisfy

$$\{ \mathscr{J}_\pm(\sigma), \overline{P}(\sigma') \} = \mathscr{J}_\pm(\sigma') \delta'_{\sigma\sigma'}.$$

In order not to spoil the first class nature of $T'_{\pm\pm}$ and the time flow generated by the Hamiltonian we add a term F that is at most quadratic in the constraints. Define the extended stress tensor

$$\overline{T}_{++} = T'_{++} - \left\langle I_1^{(0)} \varphi^{(0)} \right\rangle + \frac{1}{2} F, \quad \overline{T}_{--} = T'_{--} + \frac{1}{2} F.$$

Then,

$$\overline{P} = \overline{T}_{++} - \overline{T}_{--}, \quad \overline{H} = \overline{T}_{++} + \overline{T}_{--}.$$

If we choose

$$F = -2\alpha z_+^4 \left\langle \varphi^{(1)} \varphi^{(3)} \right\rangle$$

something remarkable happens: $\overline{H}$ becomes the boundary contribution to the canonical Hamiltonian of a Chern-Simons Theory on the disc!.
Once we have explicit expressions for $\overline{T}_{\pm\pm}$ we can construct our strongly flat extended Lax connection.

The extended Lax pair

To find $\overline{\mathscr{L}}_{\pm}(z)$, we need to compute the action of $\overline{P}_{\pm} \equiv \overline{T}_{\pm\pm}$ on the Kac-Moody currents. We find that

$$\Big\{ \mathscr{J}_{+}, \int_{S^1} d\sigma' \overline{P}_{\pm}(\sigma') \Big\} = \frac{k}{2\pi} \partial_\sigma \overline{\mathscr{L}}_{\pm}(z_{-}) + \big[\mathscr{J}_{+}, \overline{\mathscr{L}}_{\pm}(z_{-}) \big],$$
$$\Big\{ \mathscr{J}_{-}, \int_{S^1} d\sigma' \overline{P}_{\pm}(\sigma') \Big\} = -\frac{k}{2\pi} \partial_\sigma \overline{\mathscr{L}}_{\pm}(z_{+}) + \big[\mathscr{J}_{-}, \overline{\mathscr{L}}_{\pm}(z_{+}) \big],$$

where

$$\overline{\mathscr{L}}_{+}(z_{-}) = \mathscr{L}_{+}(z_{-}) + (2\pi/k)\varphi^{(0)} + \alpha z_{-}^{4} \big(\varphi^{(1)} + \varphi^{(3)} \big),$$
$$\overline{\mathscr{L}}_{-}(z_{-}) = \mathscr{L}_{-}(z_{-}) + \alpha z_{+}^{4} \big(\varphi^{(1)} + \varphi^{(3)} \big),$$
$$\overline{\mathscr{L}}_{+}(z_{+}) = \mathscr{L}_{+}(z_{+}) + \alpha \big(z_{-}^{2} \varphi^{(1)} + z_{+}^{2} \varphi^{(3)} \big),$$
$$\overline{\mathscr{L}}_{-}(z_{+}) = \mathscr{L}_{-}(z_{+}) + \alpha \big(z_{-}^{2} \varphi^{(1)} + z_{+}^{2} \varphi^{(3)} \big),$$

are extensions of $\mathscr{L}_{\pm}(z)$ but evaluated at the points $z = z_{\pm}$.

As $\overline{P}$ generate translations, the first conclusion is that we must still have

$$\overline{\mathscr{L}}_{\sigma}(z_{\mp}) = \pm\frac{2\pi}{k}\mathscr{J}_{\pm}.$$

A $\overline{\mathscr{L}}_{\sigma}(z)$ satisfying this condition is (comes from hybrid superstring)

$$\overline{\mathscr{L}}_{\sigma}(z) = f_{+}(z)\overline{\Omega}(z/z_{-})\mathscr{J}_{+} + f_{-}(z)\overline{\Omega}(z/z_{+})\mathscr{J}_{-},$$

where $f_{\pm}(z) = \alpha\left(z^4 - z_{\pm}^4\right)$ and

$$\overline{\Omega}(z) = P^{(0)} + z^{-3}P^{(1)} + z^{-2}P^{(2)} + z^{-1}P^{(3)}.$$

It obeys the Maillet bracket

$$\begin{aligned}\{\overline{\mathscr{L}}_{\sigma 1}(\sigma; z), \overline{\mathscr{L}}_{\sigma 2}(\sigma'; w)\} =& [\mathfrak{r}_{12}(z,w), \overline{\mathscr{L}}_{\sigma 1}(\sigma; z) + \overline{\mathscr{L}}_{\sigma 2}(\sigma'; w)]\delta_{\sigma\sigma'} \\ &+ [\mathfrak{s}_{12}(z,w), \overline{\mathscr{L}}_{\sigma 1}(\sigma; z) - \overline{\mathscr{L}}_{\sigma 2}(\sigma'; w)]\delta_{\sigma\sigma'} \\ &- 2\mathfrak{s}_{12}(z,w)\delta'_{\sigma\sigma'},\end{aligned}$$

The $\mathfrak{r}$ and $\mathfrak{s}$ are the anti-symmetric parts of

$$R_{12}(z,w) = -\frac{2}{z^4 - w^4} \sum_{j=0}^{3} z^j w^{4-j} C_{12}^{(j,4-j)} \varphi_\lambda^{-1}(w),$$

where $\varphi_\lambda(z)$ is the deformed twisted function

$$\varphi_\lambda(z) = \frac{2}{\alpha} . \frac{1}{(z^2 - z^{-2})^2 - (z_+^2 - z_-^2)^2}.$$

Expanding around $z = 0$ and $z = \infty$, we get

$$\overline{\mathscr{L}}_\sigma(z) = \overline{\mathscr{L}}_+(z) - \overline{\mathscr{L}}_-(z)$$

where

$$\begin{aligned} \overline{\mathscr{L}}_+(z) &= \mathscr{L}_+(z) + f_+(z)\varphi^{(0)} + \alpha z z_-^3 \varphi^{(1)} + \alpha z^3 z_- \varphi^{(3)}, \\ \overline{\mathscr{L}}_-(z) &= \mathscr{L}_-(z) + \alpha z^{-1} z_+^3 \varphi^{(3)} + \alpha z^{-3} z_+ \varphi^{(1)}. \end{aligned}$$

These satisfy all the conditions found for $\overline{\mathscr{L}}_+(z_\pm)$, $\overline{\mathscr{L}}_-(z_\pm)$. From this we easiliy find the time component as well

$$\overline{\mathscr{L}}_\tau(z) = \overline{\mathscr{L}}_+(z) + \overline{\mathscr{L}}_-(z).$$

This extended Lax pair is strongly flat because

$$\{\overline{\mathscr{L}}_{-}(z), \overline{p}_{+}\} - \{\overline{\mathscr{L}}_{+}(z), \overline{p}_{-}\} = -[\overline{\mathscr{L}}_{+}(z), \overline{\mathscr{L}}_{-}(z)],$$

where

$$\overline{p}_{\pm} = \int_{S^1} d\sigma \, \overline{T}_{\pm\pm}(\sigma).$$

We can show that

$$\left\{\overline{\mathscr{L}}_{\sigma}(\sigma; z), \overline{T}_{\pm\pm}(\sigma')\right\} = \overline{\mathscr{L}}_{\pm}(\sigma'; z)\delta'_{\sigma\sigma'} - \left[\overline{\mathscr{L}}_{\pm}(\sigma; z), \overline{\mathscr{L}}_{\sigma}(\sigma'; z)\right] \delta_{\sigma\sigma'}.$$

From this follows that the trace of the monodromy matrix is conserved. The extended Hamiltonian and momentum take the quadratic form

$$\begin{aligned}
\overline{H} &= \frac{k}{4\pi}\left\langle \overline{\mathscr{L}}_{\tau}(z_+)\overline{\mathscr{L}}_{\sigma}(z_+) - \overline{\mathscr{L}}_{\tau}(z_-)\overline{\mathscr{L}}_{\sigma}(z_-)\right\rangle, \\
\overline{P} &= \frac{k}{8\pi}\left\langle (\overline{\mathscr{L}}^2_{\tau}(z_+) + \overline{\mathscr{L}}^2_{\sigma}(z_+)) - (\overline{\mathscr{L}}^2_{\tau}(z_-) + \overline{\mathscr{L}}^2_{\sigma}(z_-))\right\rangle
\end{aligned}$$

precisely for the F as chosen above. The connection with CS is now more evident.

Consider the Monodromy matrix and the constraints

$$m(z) = P\exp[-\int_{S^1} d\sigma \overline{\mathscr{L}}_\sigma(\sigma; z)], \quad \Phi = (\varphi^{(i)}, \overline{T}_{\pm\pm}),$$

for $i = 1, 2, 3$. The relevant Poisson brackets are (take $\sigma \in [0, 2\pi]$)

$$\begin{aligned}
\{m(z), \phi_{\epsilon^{(0)}}\} &= [\epsilon^{(0)}(0), m(z)], \\
\{m(z), \phi_{\epsilon^{(3)}}\} &= (z_-/z)[\epsilon^{(3)}(0), m(z)] - 2\varphi_\lambda^{-1}(z) f(\epsilon^{(3)}, \varphi^{(1)}), \\
\{m(z), \phi_{\epsilon^{(1)}}\} &= z_+ z[\epsilon^{(1)}(0), m(z)] + 2z_+ z\varphi_\lambda^{-1}(z) g(\epsilon^{(1)}, \varphi^{(0)}),
\end{aligned}$$

where

$$\phi_\epsilon = \int_{S^1} d\sigma \, \langle \epsilon(\sigma)\Phi(\sigma)\rangle \, .$$

Two-fold interpretation:
I) On the constraint surface and for generic values of z, $\langle m(z)\rangle$ is first class, i.e. preserve the surface $\varphi^{(i)} \approx 0$.
II) At the poles $z = z_\pm$ of the twisting function, the constraints $\varphi^{(i)}$ generate gauge transformations strongly on $m(z_\pm)$.

There is an enhancement of gauge symmetry with generator

$$\overline{H}(\eta) = -\frac{k}{2\pi}\int_{S^1} d\sigma \left\langle \eta_+ \overline{\mathscr{L}}_\sigma(z_+) - \eta_- \overline{\mathscr{L}}_\sigma(z_-) \right\rangle .$$

It induces the full action of $PSU(2,2|4)$

$$\{m(z_\pm), H(\eta)\} = [\eta_\pm(0), m(z_\pm)].$$

For $\eta_+ = \Omega\epsilon$, $\eta_- = \epsilon$ and $\epsilon = \epsilon^{(0)} + \epsilon^{(1)} + \epsilon^{(3)}$, we recover the former result. The dressing gauge fixes just the right conjugacy classes and selects the true lambda model physical dof.

Summary:

Hamiltonian:
$\overline{h} = \frac{k}{4\pi}\int_{S^1} d\sigma \left\langle \overline{\mathscr{L}}_\tau(z_+)\overline{\mathscr{L}}_\sigma(z_+) - \overline{\mathscr{L}}_\tau(z_-)\overline{\mathscr{L}}_\sigma(z_-) \right\rangle .$
Gauge generator:
$\overline{H}(\eta) = -\frac{k}{2\pi}\int_{S^1} d\sigma \left\langle \eta_+ \overline{\mathscr{L}}_\sigma(z_+) - \eta_- \overline{\mathscr{L}}_\sigma(z_-) \right\rangle .$
Kac-Moody algebra :
$\{\overline{\mathscr{L}}_{\sigma 1}(\sigma; z_\pm), \overline{\mathscr{L}}_{\sigma 2}(\sigma'; z_\pm)\} = \pm\frac{2\pi}{k}([C_{12}, \overline{\mathscr{L}}_{\sigma 2}(\sigma'; z_\pm)]\delta_{\sigma\sigma'} + C_{12}\delta'_{\sigma\sigma'}).$

Chern-Simons theory

Conside the following CS action on the solid cylinder

$$S_{CS} = S_{(+)} + S_{(-)},$$

where ($\overline{k} = \pm k$, opposite levels)

$$S_{\overline{k}} = \frac{\overline{k}}{4\pi} \int_{D\times\mathbb{R}} d\tau \left\langle -A\partial_\tau A + 2A_\tau F \right\rangle - \frac{\overline{k}}{4\pi} \int_{\partial D\times\mathbb{R}} \left\langle A_\tau A \right\rangle .$$

Above, $A = A_i dx^i$, $i = 1, 2$ is a gauge field on the dics and A_τ is a Lagrange multiplier. Then, we have two sets $A_{(+)i}, A_{(+)\tau}$ and $A_{(-)i}, A_{(-)\tau}$ of fields valued on $\mathfrak{psu}(2,2|4)$.

The equations of motion are:

On the bulk: $F_{ij} = 0, \quad \partial_\tau A_i = D_i A_\tau$.

On the boundary: $\left\langle \delta A_\sigma A_\tau - \delta A_\tau A_\sigma \right\rangle = 0$.

We must choose $A_\tau = A_\tau(A_\sigma)$ such that the bry eom are satisfied.

The canonical Hamiltonian is

$$h_C = -\frac{\overline{k}}{2\pi}\int_D \langle A_\tau F\rangle + \underbrace{\frac{\overline{k}}{4\pi}\int_{\partial D} d\sigma\, \langle A_\tau A_\sigma\rangle}_{\text{Recall } \overline{h} \text{ in lambda model}}.$$

The Poisson bracket is

$$\{A_{i1}(x), A_{j2}(y)\} = \frac{2\pi}{\overline{k}}\epsilon_{ij} C_{12}\delta^{(2)}_{xy}.$$

The symplectic form (Atiyah-Bott) is

$$\omega_{CS} = \frac{\overline{k}}{4\pi}\int_D \langle \delta A \wedge \delta A\rangle.$$

Using the gauge vector fields induced by gauge transformations, we find the gauge moment

$$X_\eta = (D_i\eta)^A \frac{\delta}{\delta A_i^A} \rightarrow -i_{X_\eta}\omega_{CS} = \delta H(\eta),$$

where

$$H(\eta) = \frac{\overline{k}}{2\pi}\int_D \langle \eta F \rangle \quad \underbrace{-\frac{\overline{k}}{2\pi}\int_{\partial D} d\sigma \, \langle \eta A_\sigma \rangle}_{\text{Recall } \overline{H}(\eta) \text{ in lambda model}} \quad .$$

We now perform a symplectic reduction to the reduced space of flat connections $F = 0$. The reduced symplectic form is the pull-back

$$\begin{aligned} \omega_r &= \omega_{CS}\big|_{A=-d\Psi\Psi^{-1}} \\ &= -\frac{\overline{k}}{4\pi}\int_{\partial D} d\sigma \left\langle \delta A_\sigma \wedge D_\sigma^{-1} \delta A_\sigma \right\rangle . \end{aligned}$$

Using the gauge vector fields induced by gauge transformations, we find the boundary gauge moment

$$X_\eta = (D_i \eta)^A \frac{\delta}{\delta A_i^A} \rightarrow -i_{X_\eta}\omega_r = \delta \overline{H}(\eta),$$

where

$$\overline{H}(\eta) = -\frac{\overline{k}}{2\pi}\int_{\partial D} d\sigma \, \langle \eta A_\sigma \rangle \, .$$

The moment algebra is

$$\{\overline{H}(\eta), \overline{H}(\eta')\} = \frac{\overline{k}}{2\pi} \int_{\partial D} d\sigma d\sigma' \left\langle (C_{12}\delta'_{\sigma\sigma'} + [C_{12}, A_{\sigma 2}(\sigma')]\delta_{\sigma\sigma'})(\eta \otimes \eta') \right\rangle .$$

Equivalently, the Poisson algebra at ∂D is the KM algebra (see ω_r)

$$\{A_{\sigma 1}(\sigma), A_{\sigma 2}(\sigma')\} = \frac{2\pi}{\overline{k}}([C_{12}, A_{\sigma 2}(\sigma')]\delta_{\sigma\sigma'} + C_{12}\delta'_{\sigma\sigma'}).$$

The equivalence between lambda model and Chern-simons fields at ∂D is

$$\overline{\mathscr{L}}_{\sigma}(z_{\pm}) = A_{(\pm)\sigma}, \quad \overline{\mathscr{L}}_{\tau}(z_{\pm}) = A_{(\pm)\tau}.$$

This choice solves the boundary CS eom as well.
In terms of the z-dependent field

$$A_i(z) = -\frac{k}{2\pi} f_-(z)\overline{\Omega}(z/z_+)A_{(+)i} + \frac{k}{2\pi} f_+(z)\overline{\Omega}(z/z_-)A_{(-)i}.$$

The CS Poisson bracket extends to

$$\{A_{i1}(x; z), A_{j2}(y; w)\} = -2\mathfrak{s}_{12}(z, w)\epsilon_{ij}\delta^{(2)}_{xy},$$

where $\mathfrak{s}_{12}$ is the source of the non ultralocality of the lambda model.

z-dependent Wilson loop algebra

We now compute the precursor of the algebra of the monodromy matrix prior to the symplectic reduction.
Consider the z-dependent transport matrix for a path $x^i(t') \subset D$, $t' \in [\bar{t}, t]$

$$T(t,\bar{t};z) = P\exp[-\int_{\bar{t}}^{t} dt' \frac{dx^i(t')}{dt'} A_i(x(t');z)].$$

For two transport matrices associated to the paths $x^i(t') \subset D$, $t' \in [\bar{t}, t]$ and $y^i(s') \subset D$, $s' \in [\bar{s}, s]$ that intersect a a single point $x^i(\hat{s}) = y^i(\hat{s})$ we get

$$\begin{aligned} \{T(t,\bar{t};z)_1, T(s,\bar{s};w)_2\} &= -2T(t,\hat{s};z)_1 T(s,\hat{s};w)_2 \\ &\quad \times \mathfrak{s}_{12}(z,w) T(\hat{s},\bar{t};z)_1 T(\hat{s},\bar{s};w)_2. \end{aligned}$$

When they intersect at several point we sum over a discrete set of contributions.

However, if the two paths coincide we get, naively

$$\begin{aligned}\{T(t,\bar{t};z)_1, T(t,\bar{t};w)_2\} &= -2\int_{\bar{t}}^{t} ds T(t,s;z)_1 T(t,s;w)_2 \\ &\quad \times \mathfrak{s}_{12}(z,w) T(s,\bar{t};z)_1 T(s,\bar{s};w)_2.\end{aligned}$$

We now close the path into a loop γ and consider the Wilson loop

$$W(\gamma;z) = P\exp[-\oint_{\gamma} dx^i A_i(x;z)].$$

The Poisson algebra is

$$\begin{aligned}\{W(\gamma;z)_1, W(\gamma;w)_2\} &= -2\int_{\bar{t}}^{t} ds T(x(t),x(s);z)_1 T(x(t),x(s);w)_2 \\ &\quad \times \mathfrak{s}_{12}(z,w) T(x(s),x(t);z)_1 T(x(s),x(t);w)_2.\end{aligned}$$

If we split

$$\mathfrak{s}_{12}(z,w) = a(z,w) \otimes b(z,w)$$

and define the Wilson loop with an "impurity" inserted at the point $x(s)$

$$W(\gamma_*^s;z) = T(x(t),x(s);z) * T(x(s),x(t);z).$$

Then, we have the quadratic algebra

$$\{W(\gamma;z)_1, W(\gamma;w)_2\} = -2\int_{\bar{t}}^{t} ds W(\gamma_a^s;z)_1 W(\gamma_b^s;w)_2.$$

An interesting particular case is found when we evaluate the single point intersection bracket at the points $z = z_\pm$

$$\begin{aligned}\{T(t,\bar{t};z_\pm)_1, T(s,\bar{s};z_\pm)_2\} &= \frac{2\pi}{\bar{k}} T(t,\hat{s};z_\pm)_1 T(s,\hat{s};z_\pm)_2 \\ &\quad \times C_{12}(z,w) T(\hat{s},\bar{t};z_\pm)_1 T(\hat{s},\bar{s};z_\pm)_2.\end{aligned}$$

This last expression is the core of the Goldman bracket, when punctures and the tensor Casimir are considered in some particular representations. An important comment is in order. $W(\gamma;z)$ above always depends on the area enclosed by γ even if we restrict to flat gauge fields. Indeed, if we use the flatness conditions at $z_\pm$

$$\partial_i A_{j(\pm)} - \partial_j A_{i(\pm)} + [A_{i(\pm)}, A_{j(\pm)}] = 0$$

simultaneously to calculate

$$F_{ij}(z) = \partial_i A_j(z) - \partial_j A_i(z) + [A_i(z), A_j(z)].$$

We find that

$$F_{ij}(z) = \varphi_\lambda^{-1}(z) X_{ij}(z),$$

where $X_{ij}(z)$ denotes a combination of commutators of the components of $A_{i(\pm)}$ that never vanishes.

There are two ways to keep Wilson loops non-trivial on the disc:

- By introducing punctures as is usual in CS theory.
- By introducing an spectral parameter z dependence, where the twisting function now plays the rôle of an obstruction.

The equivalence between lambda model and Chern-simons fields at ∂D is now

$$\overline{\mathscr{L}}_\sigma(z) = A_\sigma(z), \quad \overline{\mathscr{L}}_\tau(z) = A_\tau(z).$$

Denote $W(z) = W(\partial S; z)$. After the symplectic reduction we have that

$$W(z) \rightarrow m(z).$$

This is the monodromy matrix of the lambda model, which is conserved in time because $\overline{\mathscr{L}}_\pm(z)$ is a strongly flat Lax connection.

Final comments

The picture between Poisson brackets is the following:

$$\begin{array}{ccc}
\{A_{i1}(x;z), A_{j2}(y;w)\}_{\text{CS}} & \xrightarrow{\text{Sympl. Red.}} & \{A_{\sigma 1}(\sigma;z), A_{\sigma 2}(\sigma';w)\}_{\text{Maillet}} \\
\big\downarrow \text{Pexp} & & \big\downarrow \text{Pexp} \\
\{W(z)_1, W(w)_2\} & \xrightarrow{\text{Sympl. Red.}} & \{m(z)_1, m(w)_2\} = \text{Unknown.}
\end{array}$$

Conclusions

i) The theory living on the boundary of the Chern-Simons theory is the lambda model. Similar to the ordinary CS/WZW connection.
ii) We can bypass the non-ultralocality of the lambda models at the cost of introducing two extra first class constraints $F_{(\pm)} \approx 0$ in a higher dimension theory.
ii) Under an eventual quantization, a potential advantage is to consider the quantum theory on a disc of finite size. i.e. a finite sized closed string.

... To be continued...

Thank you!