Integrable chiral Potts and tau2 model: Yang-Baxter and Onsager integrability, cyclic representations and parafermions

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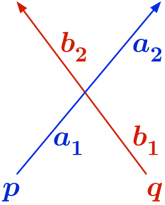
Abstract

In this talk we first introduce the integrable chiral Potts model defined by a higher-genus solution of the star-triangle (Yang-Baxter) equation. The R-matrix of this model connects with the asymmetric six-vertex model via a tau2 model as a cyclic representation in a quantum-group construction. We clarify, using some yet unpublished work, why the celebrated construction of Bazhanov and Stroganov fails for even roots-of-unity,† and how to go around it. After that we discuss some aspects of the Onsager algebra and parafermions for related quantum chains.

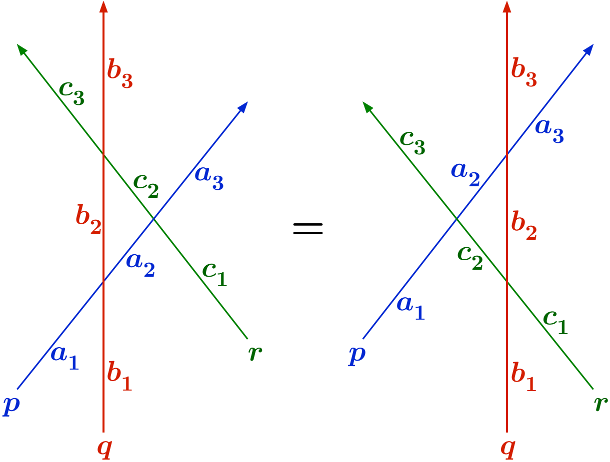
† Why Bazhanov and Stroganov, Jimbo, de Concini and Kac, Grosjean, Maillet and Niccoli, etc., only treat odd N and how to resolve the problem for even N.

$\boxed{\text{Part 1: Remarks on sl}(m,n)\text{ vertex model}}$

R-Matrix and Yang–Baxter Equation



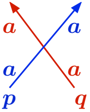
$$\omega^{b_2 a_2}_{a_1 b_1}(p,q)$$



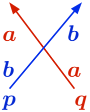
Boltzmann Weights and corresponding Yang–Baxter Equation

(= R-matrix) with rapidities p, q, r. Edge states a_2, b_2, c_2 are summed over.

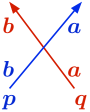
Nonzero sl(m,n) weights in fundamental representation



$$\boldsymbol{\omega}^{aa}_{aa}(p,q) = \mathcal{N} \sinh\big(\eta + \varepsilon_a(p_0 - q_0)\big)\, \frac{p_{+a}q_{-a}}{q_{+a}p_{-a}},$$
$$(a = 1, \cdots, N \equiv m + n);$$



$$\boldsymbol{\omega}^{ab}_{ba}(p,q) = \mathcal{N}\, G_{ab} \sinh\big(p_0 - q_0\big)\, \frac{p_{+a}q_{-b}}{q_{+b}p_{-a}},$$
$$(a \neq b, \quad a, b = 1, \cdots, N);$$



$$\boldsymbol{\omega}^{ba}_{ba}(p,q) = \mathcal{N}\, \mathrm{e}^{(p_0 - q_0)\mathrm{sign}(a-b)} \sinh\big(\eta\big)\, \frac{p_{+b}q_{-a}}{q_{+b}p_{-a}},$$
$$(a \neq b, \quad a, b = 1, \cdots, N).$$

($2N+1$)-component rapidities: $p = (p_{-N}, \cdots, p_{+N})$, $q = (q_{-N}, \cdots, q_{+N})$;
$\varepsilon_a = +1$ ($a = 1, \cdots m$), $\varepsilon_a = -1$ ($a = m+1, \cdots m+n$), $G_{ab}G_{ba} = 1$.

Changing the additive rapidities p_0 and q_0 to multiplicative rapidities x and y,

$$q \equiv \mathrm{e}^{2\eta}, \quad x = \mathrm{e}^{2q_0}, \quad y = \mathrm{e}^{2p_0}, \quad \mathcal{N}\,\frac{q^{1/2}}{2}\Big(\frac{y}{x}\Big)^{1/2} \equiv 1, \quad p_{\pm a} = q_{\pm a} \equiv 1, (a \neq 0),$$

we get

$$\boldsymbol{\omega}^{aa}_{aa}(p,q) = \begin{cases} 1 - q^{-1}\dfrac{x}{y}, & \text{if } \varepsilon_a = +1, \text{ for } m \text{ different } a\text{-values}, \\ \dfrac{x}{y} - q^{-1}, & \text{if } \varepsilon_a = -1, \text{ for } n \text{ different } a\text{-values}, \end{cases}$$

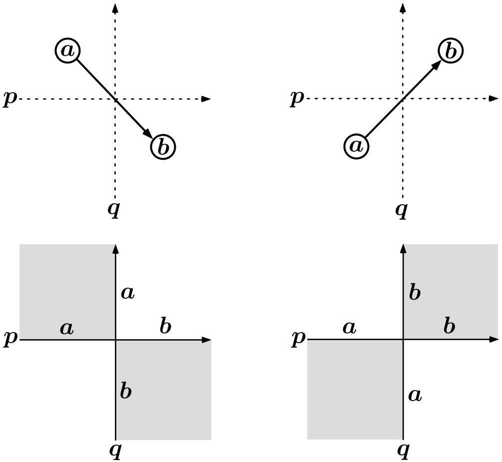
$$\boldsymbol{\omega}^{ab}_{ba}(p,q) = G_{ab}\, q^{-1/2}\big(1 - \tfrac{x}{y}\big), \quad \Longrightarrow \quad \begin{cases} 1 - \dfrac{x}{y}, & \text{if } a > b, \\ q^{-1}\big(1 - \tfrac{x}{y}\big), & \text{if } a < b, \end{cases}$$

$$\boldsymbol{\omega}^{ba}_{ba}(p,q) = \begin{cases} (1 - q^{-1})\dfrac{x}{y}, & \text{if } a > b, \\ 1 - q^{-1}, & \text{if } a < b. \end{cases}$$

If $\eta = n\pi\mathrm{i}/N$, then $q \equiv \mathrm{e}^{2\eta} = \mathrm{e}^{2n\pi\mathrm{i}/N}$, the root-of-unity case, one may try to find cyclic representations of quantum groups. The standard choice $G_{ab} \equiv 1$ leads to complications that can be resolved choosing $\boxed{G_{ab} = q^{\pm\mathrm{sign}(a-b)/2}}$, $(G_{ab}G_{ba} = 1)$, instead. Then any $\boldsymbol{\omega}^{cd}_{ab}(p,q)$ is a linear combination of $1, q^{-1}, \frac{x}{y}, q^{-1}\frac{x}{y}$ only!

Part 2: Integrable chiral Potts model

Integrable chiral Potts model Boltzmann weights



$\boldsymbol{W_{pq}(a\text{–}b)}$ $\qquad$ $\boldsymbol{\overline{W}_{pq}(a\text{–}b)}$

$p = (a_p, b_p, c_p, d_p),$

$q = (a_q, b_q, c_q, d_q).$

Boltzmann weights:

$$\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j},$$

$$\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}.$$

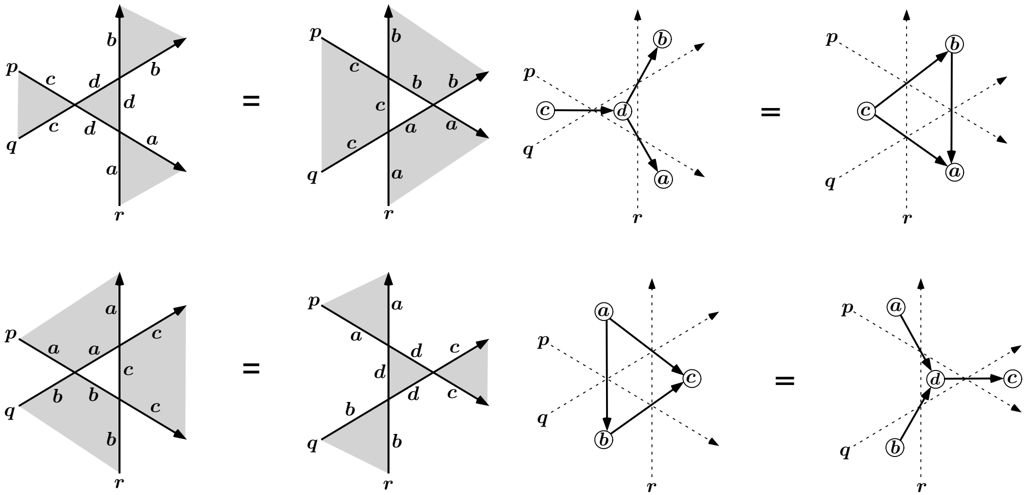
Chiral Potts curve:

$$a_p^N + k' b_p^N = k\, d_p^N,$$

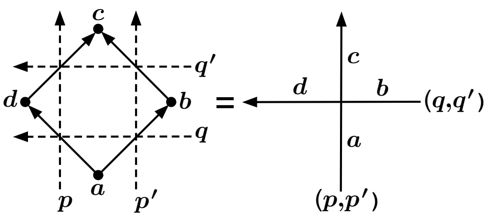
$$k' a_p^N + b_p^N = k\, c_p^N,$$

$$k^2 + k'^2 = 1, \quad \omega = \mathrm{e}^{2\pi \mathrm{i}/N}.$$

Checkerboard Yang–Baxter Equation vs Star-Triangle Equation



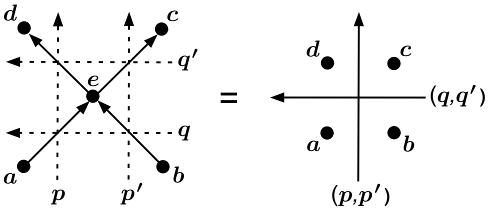
The Diamond and the Star of Four Boltzmann Weights



The shading can now be forgotten.

Bazhanov and Stroganov used this map to relate chiral Potts with the six-vertex model for $N=\text{odd}$.

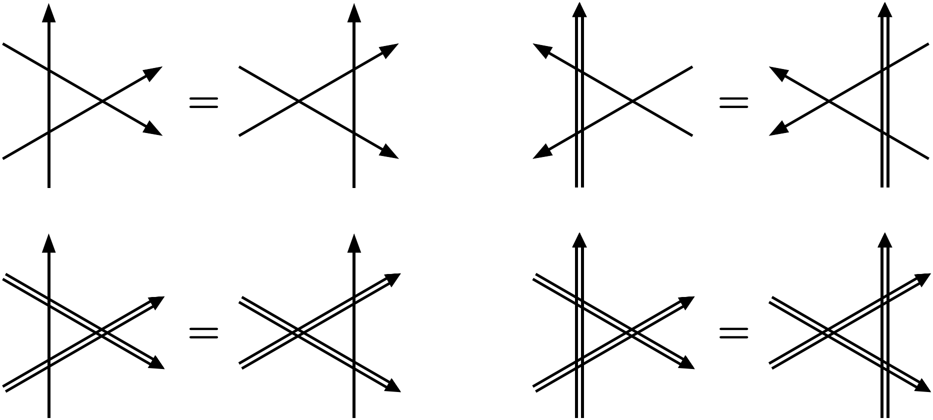
J. Stat. Phys. **59**, 799–817 (1990).



Baxter, Bazhanov and Perk used this instead to relate chiral Potts with the six-vertex model for all N. The τ_2 model and six-vertex model differ from Bazhanov–Stroganov's.

Int. J. Mod. Phys. B **4**, 803–870 (1990).

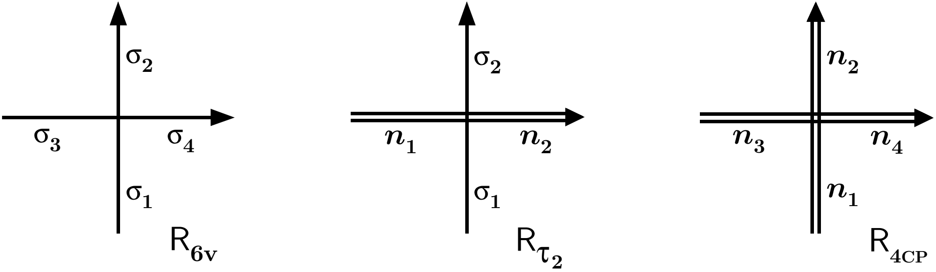
The Succession of Four Yang–Baxter Equations



Single rapidity line: spin-$\frac{1}{2}$ representation of $\mathrm{U}_q(\widehat{\mathfrak{sl}}(2,\mathbb{C}))$, quantum affine SL(2).

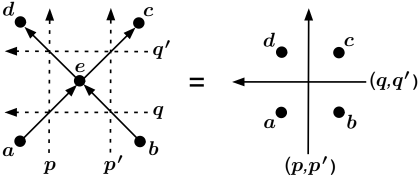
Double rapidity line: Two chiral Potts rapidities (p, p') represent a minimal cyclic representation of $\mathrm{U}_q(\widehat{\mathfrak{sl}}(2,\mathbb{C}))$, requiring q to be a root-of-unity, say $q = \omega$.

The three kinds of R-matrices of Boltzmann Weights to be Used



Here all $\sigma_i = 0, 1$, corresponding to the spin-$\frac{1}{2}$ representation.

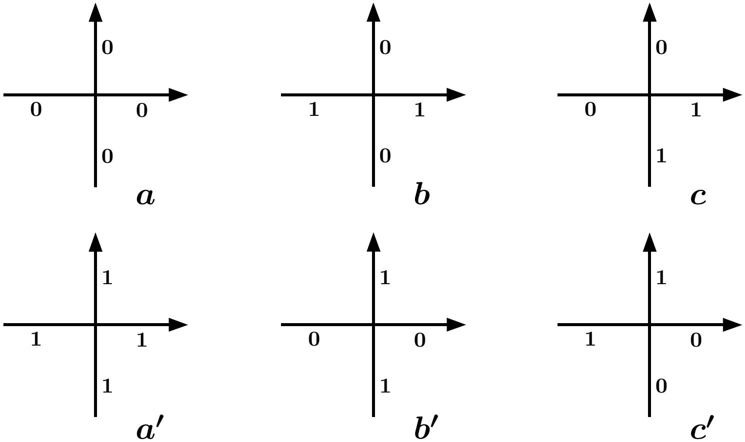
All $n_i = 0, \cdots, N-1$, i.e. $n_i \in \mathbb{Z}_N$, corresponding to the cyclic representation.



The chiral-Potts star shown on the left is also an IRF model.
In this case: $n_1 = a-b$, $n_2 = d-c$, $n_3 = a-d$, $n_4 = b-c$, (mod N), using the old Wu–Kadanoff–Wegner mapping.

$$\boxed{\text{Part 3: The odd-even } N \text{ problem in chiral Potts}}$$

The Boltzmann Weights of the Six-Vertex Model



In the symmetric six-vertex model one has $a' = a$, $b' = b$, $c' = c$. This is not the best start: Korepanov found a τ_2 model, but no chiral Potts. Different gauge choices lead to different τ_2 models that have been connected with chiral Potts.

The weights of the symmetric six-vertex model can be parametrized as

$$a = \mathcal{N}\sin(\eta + (v-u)), \quad b = \mathcal{N}\sin(v-u), \quad c = \mathcal{N}\sin(\eta),$$

with additive rapidities u and v. There is also a multiplicative parametrization:

$$q \equiv \mathrm{e}^{2\mathrm{i}\eta}, \quad x = \mathrm{e}^{2\mathrm{i}u}, \quad y = \mathrm{e}^{2\mathrm{i}v}, \quad \mathcal{C} = \mathcal{N}\,\frac{q^{1/2}}{2\mathrm{i}}\left(\frac{y}{x}\right)^{1/2},$$

so that

$$a = \mathcal{C}\left(1 - q^{-1}\frac{x}{y}\right), \quad b = \mathcal{C}\,q^{-1/2}\left(1 - \frac{x}{y}\right), \quad c = \mathcal{C}\left(1 - q^{-1}\right)\left(\frac{x}{y}\right)^{1/2}.$$

If $\eta = n\pi/N$, then $q \equiv \mathrm{e}^{2\mathrm{i}\eta} = \mathrm{e}^{2n\pi\mathrm{i}/N}$, the root-of-unity case, leading to cyclic representations of quantum groups.

However, the symmetric gauge is not a good start for the fundamental representation of sl(2) quantum: The square root $\sqrt{x/y}$ makes things ugly and it is commonly eliminated by a gauge transformation. Up to normalization $\mathcal{C}$:

$$\mathsf{R}_{\mathrm{sym}}(x,y) = \begin{pmatrix} 1-\frac{x}{y}q^{-1} & 0 & 0 & 0 \\ 0 & (1-\frac{x}{y})\,q^{-1/2} & \left(\frac{x}{y}\right)^{1/2}(1-q^{-1}) & 0 \\ 0 & \left(\frac{x}{y}\right)^{1/2}(1-q^{-1}) & (1-\frac{x}{y})\,q^{-1/2} & 0 \\ 0 & 0 & 0 & 1-\frac{x}{y}q^{-1} \end{pmatrix}$$

The $\left(\frac{x}{y}\right)^{1/2}$ and $q^{-1/2}$ cause complications especially for N even.

$$\mathsf{R}_{\mathrm{B\&S}}(x,y) = \begin{pmatrix} 1-\frac{x}{y}q^{-1} & 0 & 0 & 0 \\ 0 & (1-\frac{x}{y})\,q^{-1/2} & \frac{x}{y}(1-q^{-1}) & 0 \\ 0 & 1-q^{-1} & (1-\frac{x}{y})\,q^{-1/2} & 0 \\ 0 & 0 & 0 & 1-\frac{x}{y}q^{-1} \end{pmatrix}$$

The $q^{-1/2}$ causes complications for N even, as $(q^{-1/2})^N = -1 \neq 1$.

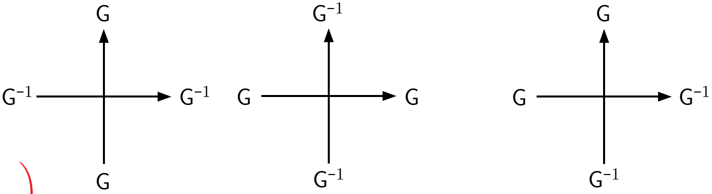
$$\mathsf{R}_{\mathrm{BBP}}(x,y) = \begin{pmatrix} 1-\frac{x}{y}q^{-1} & 0 & 0 & 0 \\ 0 & 1-\frac{x}{y} & \frac{x}{y}(1-q^{-1}) & 0 \\ 0 & 1-q^{-1} & (1-\frac{x}{y})q^{-1} & 0 \\ 0 & 0 & 0 & 1-\frac{x}{y}q^{-1} \end{pmatrix}$$

Only 1, $\frac{x}{y}$, q^{-1}, and $\frac{x}{y}q^{-1}$ show up: "smallest linear dimension".

Gauge Changes of Six-Vertex Boltzmann Weights

(sl(2) case only, not $\mathrm{sl}(m,n)$)

$$\mathsf{G}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$$



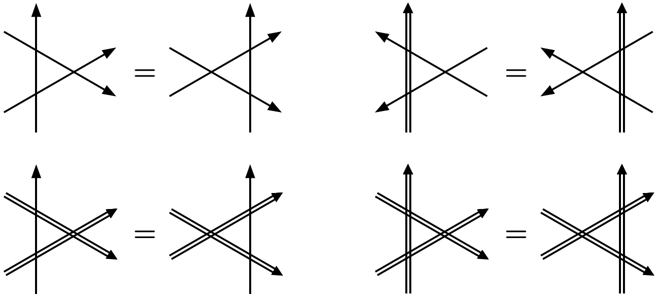
(a) (b)

A staggered gauge transform (a) with $\lambda = q^{1/8}$, can be used to connect $\mathsf{R}_{\mathrm{B\&S}}$ and $\mathsf{R}_{\mathrm{BBP}}$ in each of two different ways.

A uniform gauge transform (b) with $\lambda = (x/y)^{1/8}$ connects $\mathsf{R}_{\mathrm{sym}}$ and $\mathsf{R}_{\mathrm{B\&S}}$.

In the Baxter–Bazhanov–Perk approach there is no difficulty with even roots of unity. However, gauge transforms to the Bazhanov–Stroganov approach and then also to the Korepanov symmetric gauge, lead to complications: Two distinct τ_2 matrices arise in the $\mathsf{R}_{6\mathrm{v}}\mathsf{R}_{\tau_2}\mathsf{R}_{\tau_2}$ Yang–Baxter equation, as proposed before by Korepanov to solve the even root-of-unity problem.

The Three Different τ_2 Versions



During 1986–1987 Korepanov solved the first line using $\mathsf{R}_{\mathrm{sym}}$, giving one R_{τ_2} for $N =$ odd, while for $N =$ even his solution has two different R_{τ_2}. But he did not address the second line, so that he did not find chiral Potts.

See: J. Math. Sc. **85**, 1661-1670 (1997), St. Petersburg Math. J. **6**:2, 349-364 (1995).

Bazhanov and Stroganov were the first to address the second line starting with $\mathsf{R}_{\mathrm{B\&S}}$, the typical choice for the intertwiner of two fundamental representations of $\mathrm{U}_q(\widehat{\mathfrak{sl}}(2,\mathbb{C}))$.

However, to explicitly represent R_{τ_2} for $q=\omega\equiv\mathrm{e}^{2\pi\mathrm{i}/N}$, Bazhanov and Stroganov introduce

$$q_1=q^{(N+1)/2},\quad \text{satisfying}\quad q_1^N=1,\quad q=q_1^{-2},$$

which can only be done for $N=$odd: For $N=$even and $q=q_1^{\pm 2}$, have $q_1^N=-1$, or such q_1 is a $2N$th root of unity, leading to unresolved complications.

There is no such problem with $\mathsf{R}_{\mathrm{BBP}}$ and its R_{τ_2}. The two approaches of B&S and BBP lead to different q-Pochhammer symbols,

$$[a;q_1]_n=\prod_{k=1}^{n}(a^{-1}q_1^{k-1}-aq_1^{1-k})\quad \text{versus}\quad (a;q)_n=\prod_{k=1}^{n}(1-aq^{k-1}),$$

and q-integers,

$$[q_1]_n=\frac{q_1^n-q_1^{-n}}{q_1-q_1^{-1}}\quad \text{versus}\quad (q)_n=\frac{1-q^n}{1-q}.$$

The second forms are the usual ones of basic hypergeometrics.

Some N-state Generalization of the Pauli Matrices

$$\mathbf{X} \equiv \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \qquad \mathbf{Z} \equiv \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \omega & 0 & \dots & 0 & 0 \\ 0 & 0 & \omega^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & \omega^{N-1} \end{pmatrix},$$

$$\mathbf{Y} \equiv \begin{pmatrix} 0 & \omega^{\frac{1-N}{2}} & 0 & \dots & 0 & 0 \\ 0 & 0 & \omega^{\frac{3-N}{2}} & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \omega^{\frac{N-3}{2}} \\ \omega^{\frac{N-1}{2}} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \qquad \omega = \mathrm{e}^{2\pi \mathrm{i}/N}.$$

These matrices—generating a generalized quaternion algebra—are all unitary and

$$\mathbf{X}^N = \mathbf{Y}^N = \mathbf{Z}^N = \mathbf{1}, \qquad \mathbf{Y} = \omega^{(N-1)/2}\mathbf{X}^{-1}\mathbf{Z},$$

$$\mathbf{ZX} = \omega\mathbf{XZ}, \quad \mathbf{YX} = \omega\mathbf{XY}, \quad \mathbf{YZ} = \omega\mathbf{ZY}.$$

This is called Weyl algebra, even though it was pioneered by Sylvester in his paper on quaternions, nonions, sedenions, etc.

When $N = 2$, $\omega = -1$, so that then $\mathbf{X} = \boldsymbol{\sigma}^x$, $\mathbf{Y} = \boldsymbol{\sigma}^y$, $\mathbf{Z} = \boldsymbol{\sigma}^z$.

We can assign a copy of these operators to a site in a chain:

$$\mathbf{Z}_j = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \mathbf{1} \otimes \underset{j\text{th}}{\mathbf{Z}} \otimes \mathbf{1} \cdots \otimes \mathbf{1}, \quad \mathbf{X}_j = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \mathbf{1} \otimes \underset{j\text{th}}{\mathbf{X}} \otimes \mathbf{1} \cdots \otimes \mathbf{1},$$

$$\mathbf{Y}_j = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \mathbf{1} \otimes \underset{j\text{th}}{\mathbf{Y}} \otimes \mathbf{1} \cdots \otimes \mathbf{1},$$

so that operators on different sites commute.

These operators are used to construct the cyclic representations, but:

$\boxed{\text{Summarizing this part:}}$ Many authors end up working with "Pochhammers"

$$a^{-1}\mathbf{Z}_j^{-n/2} - a\mathbf{Z}_j^{n/2} \text{ and } a^{-1}\mathbf{X}_j^{-n/2} - a\mathbf{X}_j^{n/2},$$

starting from the Drinfeld–Jimbo choice of fundamental R-matrix. This leads to trouble, resolved for odd N choosing

$$\mathbf{Z}_j^{1/2} = \mathbf{Z}_j^{(N+1)/2} \text{ and } \mathbf{X}_j^{1/2} = \mathbf{X}_j^{(N+1)/2}.$$

With the more asymmetric R-matrix we only need

$$1 - a^2\mathbf{Z}_j^n \text{ and } 1 - a^2\mathbf{X}_j^n,$$

so that there is no odd-even problem. Also, if one sets up the quantum group starting with this modified R-matrix, one ends up with the usual Pochhammers in classical basic hypergeometric functions.

$$\boxed{\text{Part 4: Onsager algebra in quantum chain models}}$$

Cluster Ising and XY model hamiltonians, like

$$\mathcal{H}^{(\mathrm{c})} = -\sum_{j=1}^{N}\left[J_x\sigma_j^x\left(\prod_{k=j+1}^{j+n}\sigma_k^z\right)\sigma_{j+n+1}^x + J_y\sigma_j^y\left(\prod_{k=j+1}^{j+n}\sigma_k^z\right)\sigma_{j+n+1}^y + B\sigma_j^z\right],$$

should be compared with the Onsager algebra for the 2D Ising model,

$$A_n = \sum_{j=1}^{N}\sigma_j^x\left(\prod_{k=j+1}^{j+n-1}\sigma_k^z\right)\sigma_{j+n}^x,$$

$$G_n = \frac{1}{2}\mathrm{i}\sum_{j=1}^{N}\left[\sigma_j^x\left(\prod_{k=j+1}^{j+n-1}\sigma_k^z\right)\sigma_{j+n}^y + \sigma_j^y\left(\prod_{k=j+1}^{j+n-1}\sigma_k^z\right)\sigma_{j+n}^x\right].$$

As periodicity $\sigma_{j\pm N}^\alpha = \sigma_j^\alpha$, $\alpha = x, y, z$ is assumed, we have

$$A_0 = -\sum_{j=1}^{N}\sigma_j^z, \quad A_{-n} = \sum_{j=1}^{N}\sigma_j^y\left(\prod_{k=j+1}^{j+n-1}\sigma_k^z\right)\sigma_{j+n}^y.$$

Therefore, $$\boxed{\mathcal{H}^{(\mathrm{c})} = -J_xA_{n+1} - J_yA_{-n-1} + BA_0}.$$

Onsager derived the following commutation rules:

$$[A_j, A_k] = 4G_{j-k}, \quad [G_m, A_l] = 2A_{l+m} - 2A_{l-m}, \quad [G_j, G_k] = 0.$$

From these we also have "Dolan–Grady relations"

$$[A_j, [A_j, [A_j, A_k]]] = 16[A_j, A_k], \quad [A_j, [A_j, G_k]] = 16G_k.$$

These relations also apply to the superintegrable chiral Potts chain discovered by von Gehlen and Rittenberg. However, Onsager's lattice periodicity relations

$$\begin{aligned}
&A_{n\pm N} = -PA_n = -A_nP, \qquad P \equiv \prod_{k=1}^{N} \sigma_k^z, \\
&G_0 = 0, \quad G_{-n} = -G_n, \quad G_{n\pm N} = -PG_n = -G_nP, \\
&A_{n\pm 2N} = A_n, \quad G_{n\pm 2N} = G_n,
\end{aligned}$$

only hold for the 2-state chiral Potts (= Ising) case.

If we fermionize (following Kaufman, 1949):

$$\Gamma_{2j-1} = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^x, \quad \Gamma_{2j} = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^y, \quad \sigma_j^z = -\mathrm{i}\Gamma_{2j-1}\Gamma_{2j},$$

satisfying

$$\Gamma_k \Gamma_l + \Gamma_l \Gamma_k = 2\delta_{kl}\mathbf{1},$$

$$c_j = \frac{1}{2}(\Gamma_{2j-1} - \mathrm{i}\Gamma_{2j}), \quad c_j^\dagger = \frac{1}{2}(\Gamma_{2j-1} + \mathrm{i}\Gamma_{2j}),$$

the Hamiltonian becomes

$$\mathcal{H}^{(\mathrm{c})} = \mathrm{i} \sum_{j=1}^{N} \Big[J_x \Gamma_{2j} \Gamma_{2j+2n+1} - J_y \Gamma_{2j-1} \Gamma_{2j+2n+2} + B \Gamma_{2j-1} \Gamma_{2j} \Big].$$

As Γ_j is not periodic mod $2N$, but periodic mod $4N$,

$$\Gamma_{j \pm 2N} = P \Gamma_j, \quad \Gamma_{j \pm 4N} = \Gamma_j, \quad P = \prod_{k=1}^{N} (-\mathrm{i}\Gamma_{2k-1}\Gamma_{2k}),$$

the Hilbert space breaks up into two sectors, on which $\mathcal{H}^{(\mathrm{c})}$ acts as either a cyclic or an anticyclic quadratic fermion operator (Kaufman, 1949).

Assuming $N = (n+1)N_1$, we relabel the operators according to

$$\Gamma^{(p)}_{2k+1} = \Gamma_{2p+2k(n+1)+1}, \quad \Gamma^{(p)}_{2k+2} = \Gamma_{2p+2k(n+1)+2},$$

with

$$p \equiv \left\lfloor (n+1) \left\{ \frac{j-1}{2(n+1)} \right\} \right\rfloor = 0, \cdots, n, \quad k \equiv \left\lfloor \frac{j-1}{2(n+1)} \right\rfloor = 0, \cdots, N_1 - 1,$$

where $\lfloor x \rfloor$ = floor of x,$\{x\}$ = fractional part of x, and

$$\Gamma^{(p)}_k \Gamma^{(q)}_l + \Gamma^{(q)}_l \Gamma^{(p)}_k = 2\delta_{pq}\delta_{kl}\mathbf{1}.$$

We find

$$\mathcal{H}^{(\mathrm{c})} = \sum_{p=0}^{n} \mathcal{H}^{(p)}, \quad \mathcal{H}^{(p)} = \mathrm{i} \sum_{k=1}^{N_1} \left[J_x \Gamma^{(p)}_{2k} \Gamma^{(p)}_{2k+1} - J_y \Gamma^{(p)}_{2k-1} \Gamma^{(p)}_{2k+2} + B \Gamma^{(p)}_{2k-1} \Gamma^{(p)}_{2k} \right].$$

We can now define

$$\sigma_j^{z(p)} = -\mathrm{i}\Gamma_{2j-1}^{(p)}\Gamma_{2j}^{(p)},$$

$$\sigma_j^{x(p)} = \left(\prod_{k=1}^{j-1}\sigma_k^{z(p)}\right)\Gamma_{2j-1}^{(p)}, \quad \sigma_j^{y(p)} = \left(\prod_{k=1}^{j-1}\sigma_k^{z(p)}\right)\Gamma_{2j}^{(p)},$$

so that

$$\mathcal{H}^{(p)} = -\sum_{j=1}^{N_1}\Big[J_x\sigma_j^{x(p)}\sigma_{j+1}^{x(p)} + J_y\sigma_j^{y(p)}\sigma_{j+1}^{y(p)} + B\sigma_j^{z(p)}\Big], \quad p = 0, \cdots, n.$$

Thus $\mathcal{H}^{(c)}$ is decoupled into $n+1$ commuting XY chains, with identical couplings J_x, J_y and field B, thus factorizing $\exp(\beta\mathcal{H}^{(c)})$.† Thus the partition function and the spin correlations factorize (in the thermodynamic limit for the closed chain). Some factors may be zero or one.

If $B=0$ one has $2(n+1)$ transverse-field Ising chain factors.

† For the closed chain one has to deal with the odd and even fermion sectors as usual.

For more details see arXiv:1710.03384. In this talk there is time only for the simplest case, the $N \to \infty$ equilibrium bulk correlation functions of $\mathcal{H}^{(\mathrm{c})}$,

$$Z^{(\mathrm{c})}(k) = \langle \sigma_j^z \sigma_{j+k}^z \rangle, \quad M_z^{(\mathrm{c})} = \langle \sigma_j^z \rangle,$$

in terms of the corresponding $Z(k)$ and M_z for the standard XY chains $\mathcal{H}^{(p)}$. As

$$\sigma_j^z \sigma_{j+k}^z = \sigma_{k_1}^{z(p_1)} \sigma_{k_2}^{z(p_2)},$$

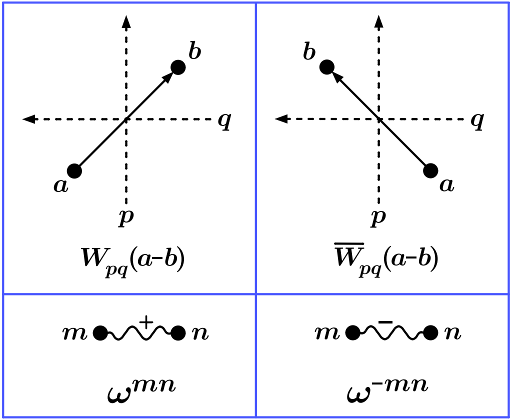
with $p_1 = p_2$ only if k is a multiple of $n+1$, we find

$$Z^{(\mathrm{c})}\big(k(n+1)\big) = Z(k), \quad \text{but } Z^{(\mathrm{c})}(m) = M_z^2, \text{ if } m \not\equiv 0 \bmod n+1.$$

Now we have only one or two factors remaining, as the other n or $n-1$ factors are trivially equal to one.

$\boxed{\text{Part 5: Remarks on (free) parafermions}}$

Chiral Potts Boltzmann Weights and Discrete Fourier



Here we forget some normalization factors $1/N$ or $1/\sqrt{N}$ with the discrete Fouriers.

States of internal vertices are summed over.

$$p = (a_p, b_p, c_p, d_p),$$

$$q = (a_q, b_q, c_q, d_q).$$

Boltzmann weights:

$$\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j},$$

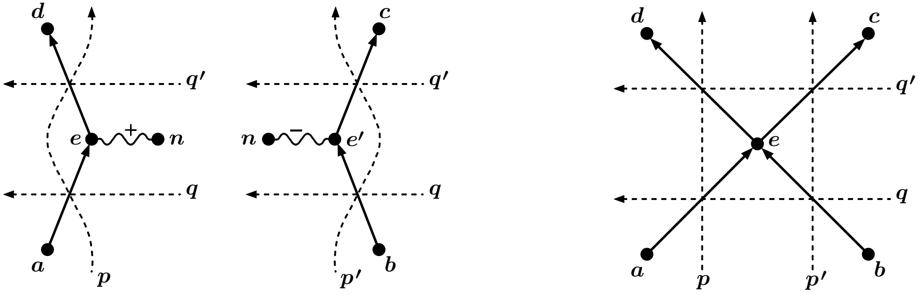
$$\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}.$$

Chiral Potts curve:

$$a_p^N + k' b_p^N = k\, d_p^N,$$

$$k' a_p^N + b_p^N = k\, c_p^N,$$

$$k^2 + k'^2 = 1, \quad \omega = \mathrm{e}^{2\pi \mathrm{i}/N}.$$



$V_{pqq'}(a,d;n)$ $V_{p'q'q}(-c,-b;n)$ $U_{pp'qq'}(a,b,c,d)$

$$U_{pp'qq'}(a,b,c,d) = \sum_{n=0}^{N-1} V_{pqq'}(a,d;n)V_{p'q'q}(-c,-b;n),$$

$$V_{pqq'}(a,d;n) = \sum_{e=0}^{N-1} \omega^{ne} W_{pq}(a-e)\overline{W}_{pq'}(e-d),$$

$$V_{p'q'q}(-c,-b;n) = \sum_{e'=0}^{N-1} \omega^{-ne'} \overline{W}_{p'q}(b-e')W_{p'q'}(e'-c).$$

Note that $V_{pqq'}(a,d;n)$ appeared before in the Fourier-transformed star-triangle equation, (both in the discovery and in the proof of the chiral Potts solution),[†]

$$W_{qq'}(a-d)V_{pqq'}(a,d;n)=R^{-1}_{pqq'}V_{pq'q}(a,d;n)\widehat{\overline{W}}_{qq'}(n),\quad \widehat{\overline{W}}_{qq'}(n)\equiv\sum_{k=0}^{N-1}\omega^{nk}\overline{W}_{qq'}(k).$$

If $q=(a_q,b_q,c_q,d_q)$, $q'=(b_q,\omega^2a_q,d_q,c_q)$, (both on the chiral Potts curve !), then it is easily checked that

$$W_{qq'}(n)=\widehat{\overline{W}}_{qq'}(n)=0,\ \text{ if } n\neq 0,1 \bmod N,$$

so that one has the triangularity conditions[‡]

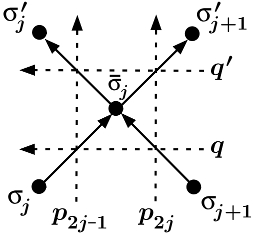
$$\begin{cases} V_{pqq'}(a,d;n)=0, & \text{if } a-d=0 \text{ or } 1,\text{ but } n\neq 0,1;\\[2ex] V_{pq'q}(a,d;n)=0, & \text{if } n=0 \text{ or } 1,\text{ but } a-d\neq 0,1;\\[2ex] U_{pp'qq'}(a,b,c,d)=0, & \text{if } a-d=0 \text{ or } 1,\text{ but } b-c\neq 0,1. \end{cases}$$

---

† H. Au-Yang, B.M. McCoy, J.H.H. Perk, S. Tang, M.-L. Yan, Phys. Lett. A **123**, 219–223 (1987), eq. (19); H. Au-Yang, J.H.H. Perk, Adv. Stud. Pure Math. **4**, 57–94 (1989), appendix.

‡ R.J. Baxter, V.V. Bazhanov, J.H.H. Perk, Int. J. Mod. Phys. B **4**, 803–870 (1990), eq. (2.26). (The chiral Potts curve makes it nontrivial. Also, one needs to use the explicit form of $R_{pqq'}$.)

If one assumes periodic boundary conditions in the horizontal direction, then the transfer matrix becomes block diagonal: In the first block each spin is 0 or 1 higher than the one above it, giving the τ_2 model, while in the second block it is $2, \cdots, N-1 \bmod N$ higher, resulting in a τ_{N-2} model.



For the first block, one has the "IRF" transfer matrix

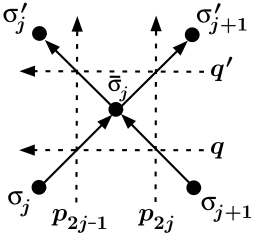
$$\boldsymbol{\tau}_2(t)_{\sigma,\sigma'} = \prod_{j=0}^{L} W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}', \sigma_j')$$

with $(\sigma_{L+1} \equiv \sigma_0,\ \sigma_{L+1}' \equiv \sigma_0')$, and where leaving out some common factors of the weights at site j, and with the (q, q') collapsed to a single variable t, Baxter found

$$W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j) = b_{2j-1}b_{2j} - \omega^{\sigma_j - \sigma_{j+1}+1} t c_{2j-1}c_{2j},$$
$$W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j - 1) = -\omega t d_{2j-1}b_{2j} + \omega^{\sigma_j - \sigma_{j+1}+1} t a_{2j-1}c_{2j},$$
$$W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j) = b_{2j-1}d_{2j} - \omega^{\sigma_j - \sigma_{j+1}+1} c_{2j-1}a_{2j},$$
$$W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j - 1) = -\omega t d_{2j-1}d_{2j} + \omega^{\sigma_j - \sigma_{j+1}+1} a_{2j-1}a_{2j}.$$

It is easily checked that these $\boldsymbol{\tau}_2(t)$ commute, even if the $p_j = (a_j, b_j, c_j, d_j)$ do not lie on the chiral Potts curve. (But connecting with chiral Potts it is needed.)

Periodic and Open Transfer Matrices



Repeat this unit $L+1$ times, with $j = 0, \cdots, L$, to make the transfer matrix with periodic boundary conditions and column-dependent rapidities p_j.

To get the τ_2 open boundary condition case, Baxter made a special choice for p_{2L} and $p_{-1} \equiv p_{2L+1}$, which affects W_L and W_0.

Look at these two weights more carefully:

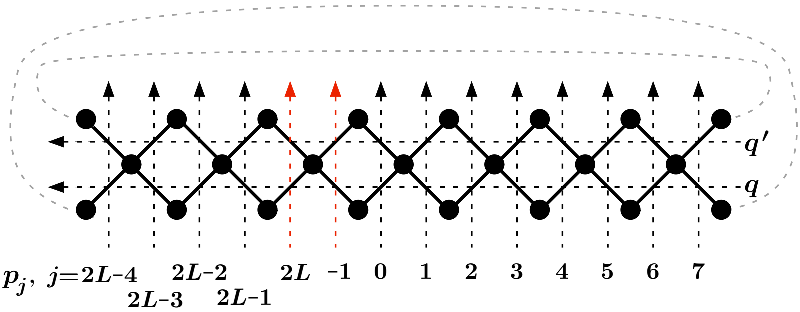
$$W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j) = b_{2j-1}b_{2j} - \omega^{\sigma_j - \sigma_{j+1}+1} t c_{2j-1}c_{2j},$$

$$W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j - 1) = -\omega t d_{2j-1}b_{2j} + \omega^{\sigma_j - \sigma_{j+1}+1} t a_{2j-1}c_{2j},$$

$$W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j) = b_{2j-1}d_{2j} - \omega^{\sigma_j - \sigma_{j+1}+1} c_{2j-1}a_{2j},$$

$$W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j - 1) = -\omega t d_{2j-1}d_{2j} + \omega^{\sigma_j - \sigma_{j+1}+1} a_{2j-1}a_{2j}.$$

We see that a lot disappears if we set $p_{2L} = (0, b_{2L}, 0, 0)$ and $p_{-1} = (0, b_{-1}, 0, 0)$.



More precisely, setting $a_{-1} = d_{-1} = c_{-1} = c_{2L} = a_{2L} = d_{2L} = 0$, one finds

$$
\begin{aligned}
W_0(\sigma_0,\sigma_1,\sigma_1,\sigma_0) &= b_0, & W_0(\sigma_0,\sigma_1,\sigma_1,\sigma_0-1) &= 0,\\
W_0(\sigma_0,\sigma_1,\sigma_1-1,\sigma_0) &= d_0, & W_0(\sigma_0,\sigma_1,\sigma_1-1,\sigma_0-1) &= 0,\\
W_L(\sigma_L,\sigma_0,\sigma_0,\sigma_L) &= b_{2L-1}, & W_L(\sigma_L,\sigma_0,\sigma_0-1,\sigma_L) &= 0,\\
W_L(\sigma_L,\sigma_0,\sigma_0,\sigma_L-1) &= -\omega t d_{2L-1}, & W_L(\sigma_L,\sigma_0,\sigma_0-1,\sigma_L-1) &= 0.
\end{aligned}
$$

This means that $\sigma_0 = \sigma_0'$ and that no weight depends on the value of σ_0. Also, σ_L and σ_1 are now uncorrelated: Free boundaries with boundary couplings.

From BBP, we have the functional equations

$$\boldsymbol{\tau}_{j+1}(t) = \boldsymbol{\tau}_j(t)\boldsymbol{\tau}_2(\omega^{j-1}t) - z(\omega^{j-1}t)\mathcal{X}\boldsymbol{\tau}_{j-1},$$
$$\boldsymbol{\tau}_{N+1} = z(\omega t)\mathcal{X}\boldsymbol{\tau}_{N-1} + [\alpha(\lambda_q) + \alpha(1/\lambda_q)]\mathbf{1},$$

with $\mathcal{X}$ the spin shift operator, $\mathcal{X}^N = \mathbf{1}$ and $z(t) \equiv 0$ for the open case.[‡] Next, as the weights are linear in t and W_0 does not depend on t now, the transfer matrix $\boldsymbol{\tau}_2(t)$ is a polynomial of degree L,

$$\boldsymbol{\tau}_2(t) = \sum_{m=0}^{L} (\omega t)^m \boldsymbol{\tau}_{2,m}, \quad \boldsymbol{\tau}_{2,0} = \boldsymbol{\tau}_2(0) = A_0\mathbf{1}, \quad A_0 \equiv \prod_{\ell=0}^{2L-1} b_\ell.$$

Therefore, from the functional equations,

$$\boldsymbol{\tau}_2(t)\boldsymbol{\tau}_2(\omega t)\cdots\boldsymbol{\tau}_2(\omega^{N-1}t) = A_0^N\mathbf{1}\prod_{j=1}^{L}(1 - r_j^N t^N),$$

which is a polynomial in t^N, as this is invariant under $t \to \omega t$. Also, $\mathbf{1}$ is the unit matrix of dimension N^{L+1}, or N^L, as σ_0 has become irrelevant.

‡ See also R.J. Baxter, J. Stat. Phys. **117** (2004) 1–25 for more discussion.

The zeros of this polynomial are $\omega^k r_j$, $(k=0,\cdots,N-1;\, j=1,\cdots,L)$, satisfying

$$s_0 r_j^{NL} + s_1 r_j^{N(L-1)} + s_2 r_j^{N(L-2)} + \cdots + s_L = 0.$$

Thus Baxter obtained all the eigenvalues of the $\boldsymbol{\tau}_2(t)$ matrix, namely

$$\tau_2(t) = A_0 \prod_{j=1}^{L} (1 - r_j \omega^{1+p_j} t), \quad 0 \le p_j \le N-1, \quad 1 \le j \le L.$$

Assuming all $b_\ell \neq 0$, we can expand

$$t\frac{d}{dt}\ln\boldsymbol{\tau}_2(t) = \sum_{m=1}^{\infty} (\omega t)^m \,\mathcal{H}^{(m)}, \quad \boldsymbol{\tau}_2(t) = A_0 \exp\left(\sum_{m=1}^{\infty} \frac{(\omega t)^m}{m}\,\mathcal{H}^{(m)}\right),$$

giving the higher Hamiltonians $\mathcal{H}^{(m)}$ and $\mathcal{H} = \mathcal{H}^{(1)} = A_0^{-1}\boldsymbol{\tau}_{2,1}$ Consequently, we also have their NL eigenvalues,

$$-\mathcal{H}^{(m)}|p_1,\cdots,p_L\rangle = \sum_{j=1}^{L} (r_j \omega^{p_j})^m |p_1,\cdots,p_L\rangle,$$

with $|p_1,\cdots,p_L\rangle$ denoting the corresponding eigenvector.

Hamiltonian in Generalized Pauli Matrices

$$\begin{aligned}
\mathcal{H} = &- \sum_{j=1}^{L} \sum_{k=j}^{L} \omega^{k-j+(N-1)/2} \frac{d_{2j-2}}{b_{2j-2}} \left(\prod_{\ell=2j-1}^{2k-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2k-1}}{b_{2k-1}} \mathbf{Z}_j \left(\prod_{\ell=j}^{k-1} \mathbf{X}_\ell \right) \mathbf{Y}_k^{-1} \\
&+ \sum_{j=1}^{L-1} \sum_{k=j+1}^{L} \omega^{k-j-1} \frac{c_{2j-1}}{b_{2j-1}} \left(\prod_{\ell=2j}^{2k-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2k-1}}{b_{2k-1}} \mathbf{Y}_j \left(\prod_{\ell=j}^{k-1} \mathbf{X}_\ell \right) \mathbf{Y}_k^{-1} \\
&- \sum_{j=1}^{L-1} \sum_{k=j}^{L-1} \omega^{k-j-(N+1)/2} \frac{c_{2j-1}}{b_{2j-1}} \left(\prod_{\ell=2j}^{2k-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2k}}{b_{2k}} \mathbf{Y}_j \left(\prod_{\ell=j}^{k} \mathbf{X}_\ell \right) \mathbf{Z}_{k+1}^{-1} \\
&+ \sum_{j=1}^{L-1} \sum_{k=j}^{L-1} \omega^{k-j} \frac{d_{2j-2}}{b_{2j-2}} \left(\prod_{\ell=2j-1}^{2k-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2k}}{b_{2k}} \mathbf{Z}_j \left(\prod_{\ell=j}^{k} \mathbf{X}_\ell \right) \mathbf{Z}_{k+1}^{-1}.
\end{aligned}$$

For the special case $N = 2$, after rotating $\mathbf{Z}_\ell \to \boldsymbol{\sigma}_\ell^x$, $\mathbf{X}_\ell \to -\boldsymbol{\sigma}_\ell^z$ and $\mathbf{Y}_\ell \to \boldsymbol{\sigma}_\ell^y$, we recognize a generalized XY-model, like the spin-chain Hamiltonian that Suzuki introduced to commute with the transfer matrix of the dimer model.

Hamiltonian in Parafermions

We define the basic parafermions as (generalized Jordan–Wigner transform)

$$\psi_{2j-2} = \Bigg(\prod_{\ell=1}^{j-1} \mathbf{X}_\ell\Bigg)\mathbf{Z}_j^{-1}, \quad \psi_{2j-1} = \Bigg(\prod_{\ell=1}^{j-1} \mathbf{X}_\ell\Bigg)\mathbf{Y}_j^{-1}, \quad \psi_0 = \boldsymbol{\Gamma}_0 = \mathbf{Z}_1^{-1},$$

for $1 \leqslant j \leqslant L$. From the commutation relations of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, it follows that

$$\psi_j \psi_k = \omega^{-1} \psi_k \psi_j \quad \text{for } j < k, \qquad \psi_j^N = \mathbf{1}.$$

The Hamiltonian may be expressed in terms of these parafermions as†

$$\begin{aligned}
\mathcal{H} = &- \sum_{j=1}^{L} \sum_{m=j}^{L} \omega^{m-j+(N-1)/2} \Bigg(\prod_{\ell=2j-1}^{2m-2} \frac{a_\ell}{b_\ell}\Bigg) \frac{d_{2j-2} d_{2m-1}}{b_{2j-2} b_{2m-1}} \psi_{2j-2}^{-1} \psi_{2m-1} \\
&- \sum_{j=1}^{L-1} \sum_{m=j}^{L-1} \omega^{m-j} \Bigg[\omega^{-(N+1)/2} \Bigg(\prod_{\ell=2j}^{2m-1} \frac{a_\ell}{b_\ell}\Bigg) \frac{c_{2j-1} c_{2m}}{b_{2j-1} b_{2m}} \psi_{2j-1}^{-1} \psi_{2m} \\
&- \Bigg(\prod_{\ell=2j-1}^{2m-1} \frac{a_\ell}{b_\ell}\Bigg) \frac{d_{2j-2} c_{2m}}{b_{2j-2} b_{2m}} \psi_{2j-2}^{-1} \psi_{2m} - \Bigg(\prod_{\ell=2j}^{2m} \frac{a_\ell}{b_\ell}\Bigg) \frac{c_{2j-1} d_{2m+1}}{b_{2j-1} b_{2m+1}} \psi_{2j-1}^{-1} \psi_{2m+1}\Bigg].
\end{aligned}$$

† The special Baxter case studied by Fendley follows setting all a_l's zero.

The Fendley–Baxter Suggestion

Define recursively

$$\mathbf{\Gamma}_0 = \mathbf{Z}_1^{-1}, \quad \mathbf{\Gamma}_{j+1} = (\omega^{-1}-1)^{-1}(\mathcal{H}\mathbf{\Gamma}_j - \mathbf{\Gamma}_j\mathcal{H}), \quad (j \geqslant 0),$$

Using $\mathbf{\Gamma}_0 = \boldsymbol{\psi}_0$, it is straightforward to show that

$$\mathbf{\Gamma}_1 = \frac{d_0}{b_0}\left[\sum_{m=1}^{L}\omega^{m+(N-1)/2}\left(\prod_{\ell=1}^{2m-2}\frac{a_\ell}{b_\ell}\right)\frac{d_{2m-1}}{b_{2m-1}}\boldsymbol{\psi}_{2m-1} - \sum_{m=1}^{L-1}\omega^{m}\left(\prod_{\ell=1}^{2m-1}\frac{a_\ell}{b_\ell}\right)\frac{c_{2m}}{b_{2m}}\boldsymbol{\psi}_{2m}\right],$$

which is rather complicated. Nevertheless, we can easily show

$$\mathbf{\Gamma}_0\mathbf{\Gamma}_1 = \omega^{-1}\mathbf{\Gamma}_1\mathbf{\Gamma}_0.$$

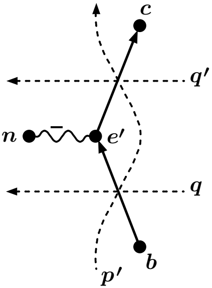
Based on numerical evidence, Baxter found that the infinite sequence of the $\mathbf{\Gamma}_j$ truncates, as he conjectured that the $\mathbf{\Gamma}$ matrices satisfy the equation

$$s_0\mathbf{\Gamma}_{NL+j} + s_1\mathbf{\Gamma}_{N(L-1)+j} + \cdots + s_L\mathbf{\Gamma}_j = 0, \quad \text{for } j = 0,$$

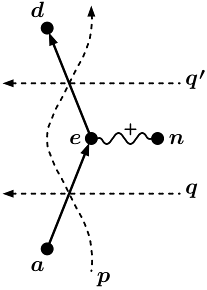
with the same coefficients s_ℓ as defined earlier in

$$\boldsymbol{\tau}_2(t)\boldsymbol{\tau}_2(\omega t)\cdots\boldsymbol{\tau}_2(\omega^{N-1}t) = (s_0t^{NL} + s_1t^{N(L-1)} + s_2t^{N(L-2)} + \cdots + s_L)\mathbf{1}.$$

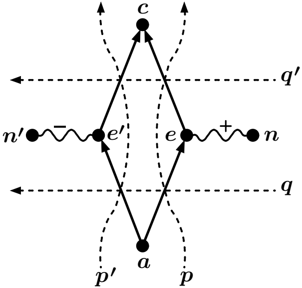
If the conjecture holds for $j = 0$, then by recurrence also for all $j > 0$. It has been proved using using the partially Fourier transformed vertex model weights $\mathsf{S}^{(\mathrm{pf})}$. (See section 4 of H. Au-Yang and J.H.H. Perk, J. Phys. A **47** (2014) 315002.)



$V_{p'q'q}(-c,-b;n)$

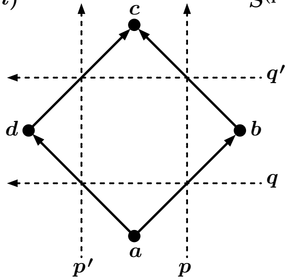


$V_{pqq'}(a,d;n)$



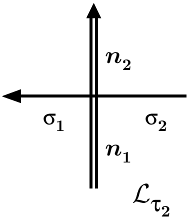
$S^{(\mathrm{pf})}{}_{p'q'q}(a,n,c,n')$

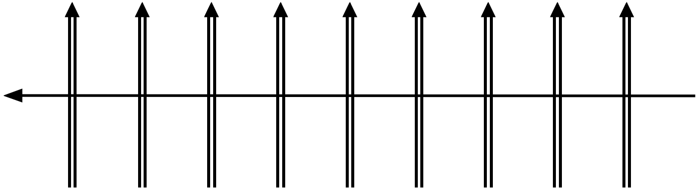
The partial Fourier gauge transform (pf) cancels out in the row-to-row transfer matrix, giving the diamond:



$S_{p'pq'q}(a,b,c,d)$

As before we set $q=(a_q,b_q,c_q,d_q)$, $q'=(b_q,\omega^2 a_q,d_q,c_q)$, so that $\mathsf{S}^{(\mathrm{pf})}$ becomes $\mathcal{L}_{\tau_2}$, a τ_2 R-matrix with $\sigma_{1,2}=0,1$ and $n_{1,2}=0,1,\cdots,N-1$.





As standard in quantum inverse scattering we construct the monodromy matrix using $L+1$ copies for $j=0,\cdots,L$, summing over the states on internal edges:

$$\mathcal{M}^{0,L}(t),\quad \text{where } \mathcal{M}^{m,n}(t)\equiv\prod_{j=m}^{n}\mathcal{L}_j(t)=\begin{pmatrix}\mathsf{A}^{m,n}(t) & \mathsf{B}^{m,n}(t)\\ \mathsf{C}^{m,n}(t) & \mathsf{D}^{m,n}(t)\end{pmatrix}.$$

After setting $a_{-1}=d_{-1}=c_{-1}=c_{2L}=a_{2L}=d_{2L}=0$ again, $\mathcal{L}_0=\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$,

and so that $\boxed{\boldsymbol{\tau}_2(t)=\mathsf{A}^{1,L}(t)}$ for the open boundary case on sites $1,\cdots,L$.

The Monodromy Matrices by Recurrence

$$\mathcal{M}^{m,n}(t) = \mathcal{M}^{m,k}(t)\mathcal{M}^{k+1,n}(t) = \begin{pmatrix} \mathsf{A}^{m,n}(t) & \mathsf{B}^{m,n}(t) \\ \mathsf{C}^{m,n}(t) & \mathsf{D}^{m,n}(t) \end{pmatrix},$$

$$\mathcal{M}^{j,j}(t) = \mathcal{L}_j(t) = \begin{pmatrix} \mathsf{A}^{j,j}(t) & \mathsf{B}^{j,j}(t) \\ \mathsf{C}^{j,j}(t) & \mathsf{D}^{j,j}(t) \end{pmatrix},$$

$$\begin{cases} \mathcal{L}_j(0,0) = \mathsf{A}^{j,j}(t) = b_{2j-2}b_{2j-1} - \omega t d_{2j-2}d_{2j-1}\mathbf{X}_j, \\ \mathcal{L}_j(0,1) = \mathsf{B}^{j,j}(t) = (-\omega t)\mathbf{Z}_j(b_{2j-2}c_{2j-1} - d_{2j-2}a_{2j-1}\mathbf{X}_j), \\ \mathcal{L}_j(1,0) = \mathsf{C}^{j,j}(t) = \mathbf{Z}_j^{-1}(c_{2j-2}b_{2j-1} - \omega a_{2j-2}d_{2j-1}\mathbf{X}_j), \\ \mathcal{L}_j(1,1) = \mathsf{D}^{j,j}(t) = \omega a_{2j-2}a_{2j-1}\mathbf{X}_j - \omega t c_{2j-2}c_{2j-1}, \end{cases}$$

$$\boxed{\boldsymbol{\tau}_2(t) = \mathsf{A}^{1,L}(t)}, \qquad \mathsf{A}^{m,n}(t) = \mathsf{A}^{m,k}(t)\mathsf{A}^{k+1,n}(t) + \mathsf{B}^{m,k}(t)\mathsf{C}^{k+1,n}(t).$$

The technical proofs of Baxter's conjecture just mentioned and the next one use this recurrence and the Yang–Baxter equation for the monodromy matrices $\mathcal{R}^{6\mathrm{v}}(t,t')\mathcal{M}^{m,n}(t)\mathcal{M}^{m,n}(t) = \mathcal{M}^{m,n}(t')\mathcal{M}^{m,n}(t)\mathcal{R}^{6\mathrm{v}}(t,t')$. For the details we refer to our paper,* as it would take too much time to explain it here.

* H. Au-Yang and J.H.H. Perk, J. Phys. A **47** (2014) 315002.

Rewriting Baxter's First Conjecture

We have just outlined what we needed to show that the recurrence

$$\boldsymbol{\Gamma}_0 = \mathbf{Z}_1^{-1}, \quad \boldsymbol{\Gamma}_{j+1} = (\omega^{-1} - 1)^{-1}(\mathcal{H}\boldsymbol{\Gamma}_j - \boldsymbol{\Gamma}_j\mathcal{H}), \quad (j \geqslant 0),$$

closes through

$$s_0\boldsymbol{\Gamma}_{NL+j} + s_1\boldsymbol{\Gamma}_{N(L-1)+j} + \cdots + s_L\boldsymbol{\Gamma}_j = 0, \quad \text{for } j = 0.$$

This then obviously holds for all j also. We can now rewrite

$$\boldsymbol{\Gamma}_j\mathcal{H} - \mathcal{H}\boldsymbol{\Gamma}_j = (1 - \omega^{-1})\boldsymbol{\Gamma}_{j+1} = (1 - \omega^{-1})\sum_{k=0}^{NL-1} h_{jk}\boldsymbol{\Gamma}_k = (1 - \omega^{-1})(\mathbf{H}\cdot\underline{\boldsymbol{\Gamma}})_j,$$

where $(j = 0, \cdots, NL-1)$ and

$$h_{ij} = \delta_{i,j-1}, \quad (0 \leqslant i < NL-1),$$
$$h_{NL-1,mN} = -s_{L-m}/s_0, \quad (0 \leqslant m < L), \qquad h_{NL-1,j} = 0, \ (j \neq 0 \bmod N).$$

Baxter's Second Conjecture

Baxter next conjectured:

$$t\boldsymbol{\nu}_j = \boldsymbol{\mu}_{j-1},$$

where

$$\boldsymbol{\mu}_j \equiv \boldsymbol{\Gamma}_j \boldsymbol{\tau}_2(t) - \boldsymbol{\tau}_2(t)\boldsymbol{\Gamma}_j, \quad \boldsymbol{\nu}_j \equiv \omega\boldsymbol{\Gamma}_j \boldsymbol{\tau}_2(t) - \boldsymbol{\tau}_2(t)\boldsymbol{\Gamma}_j.$$

We have proved this with the same tools in the paper just cited. Again the details are too technical to present.

Using $\boldsymbol{\Gamma}_{j+1} = (\mathbf{H} \cdot \underline{\boldsymbol{\Gamma}})_j$, we find

$$\mu_j = \boldsymbol{\Gamma}_j \boldsymbol{\tau}_2(t) - \boldsymbol{\tau}_2(t)\boldsymbol{\Gamma}_j = t\nu_{j+1} = \omega t\,(\mathbf{H} \cdot \underline{\boldsymbol{\Gamma}})_j\,\boldsymbol{\tau}_2(t) - t\,\boldsymbol{\tau}_2(t)\,(\mathbf{H} \cdot \underline{\boldsymbol{\Gamma}})_j,$$

or

$$\underline{\boldsymbol{\Gamma}} - \boldsymbol{\tau}_2(t)\,\underline{\boldsymbol{\Gamma}}\,\boldsymbol{\tau}_2(t)^{-1} = \omega t\,\mathbf{H} \cdot \underline{\boldsymbol{\Gamma}} - t\,\mathbf{H} \cdot \boldsymbol{\tau}_2(t)\,\underline{\boldsymbol{\Gamma}}\,\boldsymbol{\tau}_2(t)^{-1},$$

or

$$\boxed{\boldsymbol{\tau}_2(t)\,\underline{\boldsymbol{\Gamma}}\,\boldsymbol{\tau}_2(t)^{-1} = \frac{\mathbf{1} - \omega t\,\mathbf{H}}{\mathbf{1} - t\,\mathbf{H}} \cdot \underline{\boldsymbol{\Gamma}}},$$

first written down by Baxter. With this we can prove Baxter's final conjecture.

Diagonalization of Matrix $\mathbf{H}$ by a Vandermonde

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ * & * & * & * & * & \cdots & 0 & 0 \end{pmatrix} = \mathbf{P} \cdot \mathbf{H}_{\mathrm{d}} \cdot \mathbf{P}^{-1},$$

with in the last row $h_{NL-1,mN} = -s_{L-m}/s_0$, $\quad (0 \leqslant m < L)$, and 0 otherwise.

The eigenvalues are given by $\sum s_k \lambda^{N(L-k)} = 0$, i.e. $\lambda_{Nj+i+1} = r_j \omega^i$ seen before, and $\mathbf{P}$ is the Vandermonde matrix with columns $(\lambda_m)^k$, $(k = 0, \cdots, NL-1)$.

To deal with the inverse, we used Prony's 1795 result

$$f_m(z) = \prod_{n=1, n \neq m}^{NL} \frac{z - \lambda_n}{\lambda_m - \lambda_n} = \sum_{k=0}^{NL-1} (P^{-1})_{mk} z^k, \quad \text{satisfying} \quad f_m(\lambda_n) = \delta_{mn}.$$

Cyclic Raising Operators and Projection Operators

Baxter defined the candidate free parafermion operators

$$\widehat{\mathbf{\Gamma}}_i \equiv \sum_{j=0}^{NL-1} P_{ij}^{-1}\mathbf{\Gamma}_j, \qquad \mathcal{H}\widehat{\mathbf{\Gamma}}_j - \widehat{\mathbf{\Gamma}}_j\mathcal{H} = (\omega^{-1}-1)\lambda_j\widehat{\mathbf{\Gamma}}_j.$$

Generalizing Fendley, we also introduce the projection operators

$$\mathcal{P}_{\omega^p,k} = -\sum_{\ell=0}^{L-1}\sum_{q=0}^{N-1} P_{Nk+p,\ell N+q}^{-1}\mathcal{H}^{(\ell N+q)}.$$

Multiplying both sides with the Vandermonde, we find

$$\mathcal{H}^{(m)} = -\sum_{k=1}^{L}\sum_{p=0}^{N-1}(r_k\omega^p)^m\mathcal{P}_{\omega^p,k},$$

which all commute, so that

$$[\mathcal{P}_{\omega^p,k},\mathcal{P}_{\omega^q,\ell}] = 0.$$

Remember

$$\mathcal{H}^{(m)}|n_1,n_2,\cdots,n_L\rangle = -\sum_{k=1}^{L}(r_k\omega^{n_k})^m\,|n_1,n_2,\cdots,n_L\rangle,$$

so that we must have

$$\mathcal{P}_{\omega^p,k}|n_1,n_2,\cdots,n_L\rangle = \delta_{p,n_k}|n_1,n_2,\cdots,n_L\rangle,$$

from which the projection operator properties follow:

$$\mathcal{P}^2_{\omega^p,k} = \mathcal{P}_{\omega^p,k},\quad \mathcal{P}_{\omega^p,k}\mathcal{P}_{\omega^q,k} = \delta_{p,q}\mathcal{P}_{\omega^p,k},\quad \sum_{p=0}^{N-1}\mathcal{P}_{\omega^p,k} = \mathbf{1}.$$

Also,

$$\boldsymbol{\tau}_2(t) = A_0\prod_{k=1}^{L}\prod_{p=0}^{N-1}(\mathbf{1} - r_k\omega^{1+p}t\,\mathcal{P}_{\omega^p,k}) = A_0\prod_{k=1}^{L}\left(\mathbf{1} - \omega t\sum_{p=0}^{N-1} r_k\omega^p\,\mathcal{P}_{\omega^p,k}\right),$$

as this produces the correct eigenvalues seen before.

Proof of Commutation Relation of Cyclic Raising Operators

From

$$\mathcal{H}\widehat{\mathbf{\Gamma}}_j - \widehat{\mathbf{\Gamma}}_j\mathcal{H} = (\omega^{-1} - 1)\lambda_j\widehat{\mathbf{\Gamma}}_j, \qquad \mathcal{H} = \mathcal{H}^{(1)} = -\sum_{k=1}^{L}\sum_{p=0}^{N-1}(r_k\omega^p)\mathcal{P}_{\omega^p,k},$$

we find

$$\sum_{k=1}^{L}\sum_{p=0}^{N-1}(r_k\omega^p)[\mathcal{P}_{\omega^p,k}\widehat{\mathbf{\Gamma}}_{N\ell+q} - \widehat{\mathbf{\Gamma}}_{N\ell+q}\mathcal{P}_{\omega^p,k}] = r_\ell(\omega^{q-1} - \omega^q)\widehat{\mathbf{\Gamma}}_{N\ell+q}.$$

This implies the relation,

$$[\mathcal{P}_{\omega^p,k}\widehat{\mathbf{\Gamma}}_{N\ell+q} - \widehat{\mathbf{\Gamma}}_{N\ell+q}\mathcal{P}_{\omega^p,k}] = \delta_{k,\ell}(\delta_{p,q-1} - \delta_{p,q})\widehat{\mathbf{\Gamma}}_{N\ell+q}.$$

We used

$$\boldsymbol{\tau}_2(t)\,\underline{\widehat{\mathbf{\Gamma}}}\,\boldsymbol{\tau}_2(t)^{-1} = \frac{\mathbf{1} - \omega t\,\mathbf{H}_{\mathrm{d}}}{\mathbf{1} - t\,\mathbf{H}_{\mathrm{d}}}\cdot\underline{\widehat{\mathbf{\Gamma}}}, \qquad (1-r_\ell\omega^q\,t)\boldsymbol{\tau}_2(t)\widehat{\mathbf{\Gamma}}_{N\ell+q} = (1-r_\ell\omega^{q+1}\,t)\widehat{\mathbf{\Gamma}}_{N\ell+q}\boldsymbol{\tau}_2(t),$$

implying that $\widehat{\mathbf{\Gamma}}_{N\ell+q}$ only acts on the n_ℓ in $|n_1,\cdots,n_L\rangle$.

From this we can also conclude

$$\widehat{\mathbf{\Gamma}}_{Nk+p}\widehat{\mathbf{\Gamma}}_{Nk+p'} = 0, \quad \text{if } p' \neq p - 1 \bmod N.$$

Finally, we could prove the third conjecture of Baxter,

$$(r_k\omega^p - r_{k'}\omega^{p'+1})\widehat{\mathbf{\Gamma}}_{Nk+p}\widehat{\mathbf{\Gamma}}_{Nk'+p'} + (r_{k'}\omega^{p'} - r_k\omega^{p+1})\widehat{\mathbf{\Gamma}}_{Nk'+p'}\widehat{\mathbf{\Gamma}}_{Nk+p} = 0,$$

which gives the commutation relation between these operators. We can now create all the basis states by acting on $|0, 0, \cdots, 0\rangle$.

The eigenstates of the τ_2 model are useful—and have been used—as a starting point to study the chiral Potts model, the first model found with rapidities (spectral parameters) on a curve of high genus.

Thank you!

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