New advances in separation of variables for integrable quantum models with boundaries

Exactly Solvable Quantum Chains

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Subject related to past and current collaborations over the last 8 years with:

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Research framework: Quantum Integrable Models (IQMs)

- There exists a family of commuting conserved charges $T(\lambda)$.
- $T(\lambda)$ (transfer matrix) is written in terms of the generators of the Yang-Baxter algebra.

Motivations:

- Access to exact results in physics out of range of other approximate techniques, e.g. strongly interacting systems in statistical mechanics, condensed matter, string $\&$ gauge theories.

- Interface with beautiful mathematics as: conformal field theory, vertex algebras, quantum groups, combinatory, knot theory etc

Known results:

Hamiltonian spectrum, Partition functions, exact scattering S-matrix, critical exponents,

form factors, correlation functions and dynamical structure factors (measurable quantities),...

Existing problems: (quantum integrability does not imply exact solvability yet)

- Traditional methods do not apply to large classes of integrable quantum models.
- Quantum Separation of Variables (SOV) seems the right method to overcome these problems.

Main aims:

- To solve exactly lattice integrable quantum models (IQM) by quantum separation of variables (SOV) characterizing both their spectrum and dynamics.
- To define a microscopic approach to solve exactly 1+1 dimensional quantum field theories (QFT) by using the SOV solution of their integrable lattice regularizations.

Original state of art of Quantum Separation of Variables (SOV):

- The quantum version of SOV has been invented by E. Sklyanin (1985) and applied to some specific integrable quantum models (like Toda model and XXZ spin chains).
- SOV applied for few others integrable quantum models by some few key researchers: Gutzwiller, Kharchev, Lebedev, Babelon, Smirnov (Toda model), Babelon, Bernard and Smirnov (sine-Gordon model), Derkachov, Korchemsky and Manashov (non-compact XXX chain), Lukyanov, Bytsko and Teschner (sinh-Gordon model) etc
- Need for a systematic development and generalization of the SOV method.

Plan of the seminar:

- Quantum analogs of integrability, separation of variables and inverse scattering method[1].
- Separation of variables method[2] for 6-vertex Yang-Baxter & Reflection algebras.
- Universal characterization of spectrum & "dynamics" of integrable quantum models by SOV.
- Explicit SOV results for two classes of open quantum integrable models:

  A) Cyclic quantum models with the most general integrable boundaries: (e.g. chiral Potts)

  – Completeness of spectrum description by functional equation of Baxter's type.

  B) Quantum spin chains with the most general integrable boundaries: (XXX spin 1/2 case)

  – Completeness of spectrum description by functional equation of Baxter's type.

  – Matrix elements of local operators, first fundamental results toward the model dynamics.

- Projects.

[1] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Teor. Mat. Fiz. 40 (1979) 194.
[2] E. K. Sklyanin, Lect. Notes Phys. 226 (1985) 196.

Quantum analog of integrability and separation of variables

A definition of quantum integrability

- **A quantum model can be defined by:**

  I) Quantum space $\mathcal{H} \longleftrightarrow \mathcal{H}$ is an Hilbert space,

  II) Observables $\mathcal{O} \longleftrightarrow \mathcal{O} \in$ End$(\mathcal{H})$,

  III) Hamiltonian H of the quantum model $\longleftrightarrow$ time evolution operator $e^{-i\,H\theta/\hbar} \in$ End$(\mathcal{H})$.

- **Problems to solve for quantum models:**

  ◦ To compute eigenvalues and eigenvectors of $H \in$ End$(\mathcal{H})$.

  ◦ To compute matrix elements of observables $\mathcal{O} \in$ End$(\mathcal{H})$ on Hamiltonian eigenvectors: $\langle t'|\mathcal{O}|t\rangle$, where $\langle t'| \in \mathcal{H}^*$ is a co-eigenvector and $|t\rangle \in \mathcal{H}$ is an eigenvector of H.

- **Quantum integrability:**

  $\exists$ T$(\lambda) \in$ End$(\mathcal{H})$: i) $[\mathsf{T}(\lambda),\mathsf{T}(\lambda')]=0 \quad \forall\lambda, \lambda'$, ii) $[\mathsf{T}(\lambda), H]=0 \quad \forall\lambda \in \mathbb{C}$,

  iii) Complete quantum integrability: simplicity (non-degeneracy) of T(λ) spectrum.

  Note that T$(\lambda) \in$ End$(\mathcal{H})$ defines the one-parameter family of commuting conserved charges.

Quantum separation of variables (SOV): a definition

- Let $\mathsf{Y}_n \in \mathrm{End}(\mathcal{H})$ and $P_n \in \mathrm{End}(\mathcal{H})$ be N couples of canonical conjugate operators:

$$[\mathsf{Y}_n, \mathsf{Y}_m] = [P_n, P_m] = 0, \qquad [\mathsf{Y}_n, P_m] = \delta_{n,m}/2\pi i \;\; \forall (n,m) \in \{1,..,\mathsf{N}\}^2,$$

where $\{\mathsf{Y}_1, \ldots, \mathsf{Y}_\mathsf{N}\}$ are simultaneous diagonalizable operators with simple spectrum.

- **Definition**: Y_n are quantum separate variables for $T(\lambda)$ iff its eigenstates $|t\rangle$ have the form:

$$|t\rangle = \sum_{\text{over spectrum of } \{Y_n\}} \prod_{n=1}^{\mathsf{N}} \mathsf{Q}_t^{(n)}(y_n) \; |y_1, \ldots, y_\mathsf{N}\rangle,$$

where its eigenvalue $t(\lambda)$ and $Q_t^{(n)}(\lambda)$ are solutions of separate equations in y_n of the type

$$F_n(y_n, \frac{i}{2\pi}\frac{d}{dy_n}, t(y_n))\, Q_t^{(n)}(y_n) = 0, \quad \text{for all the } \; n \in \{1, \ldots, \mathsf{N}\}.$$

- The N quantum separate relations are the natural quantum analogue of the classical ones in the Hamilton-Jacobi's approach.

- The Hydrogen atom Hamiltonian represents one natural example of integrable quantum system to which quantum SOV applies, here the y_n are the spherical coordinates r, θ, ϕ.

Quantum description for the Hydrogen atom: integrability and separate variables

– Hamiltonian:

$$\langle r,\theta,\varphi|H=[-(\partial^2/\partial r^2)(\hbar^2 r/2m)+\mathbf{L}^2/2mr^2-e^2/r]\langle r,\theta,\varphi|,$$

$\mathbf{L}$ the angular momentum, vector differential operator in θ and φ only.

– H is integrable, $H_3=H$, $H_2=\mathbf{L}^2$, $H_1=\mathbf{L}_z$ C.S.C.O, and the separate relations read:

$$F_n(y_n,\frac{i}{2\pi}\frac{\partial}{\partial y_n},h_3(k,l),h_2(l),h_1(m))\Psi_{k,l,m}(r,\theta,\varphi)=0,\quad y_3=r,y_2=\theta,y_1=\varphi.$$

where $h_3(k,l)=E_I/(k+l)^2$, $h_2(l)=l(l+1)\hbar^2$, $h_1(m)=m\hbar$ and:

$$F_3(r,\frac{i}{2\pi}\frac{\partial}{\partial r},h_3(k,l),h_2(l))\equiv-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}r+\frac{h_2(l)}{2mr^2}-\frac{e^2}{r}-h_3(k,l),$$

$$F_2(\theta,\frac{i}{2\pi}\frac{\partial}{\partial\theta},h_2(l),h_1(m))\equiv-\hbar^2\frac{\partial^2}{\partial\theta^2}-\frac{\hbar^2}{\tan\theta}\frac{\partial}{\partial\theta}+\frac{h_1^2(m)}{\sin^2\theta}-h_2(l),\quad F_1(\varphi,\frac{i}{2\pi}\frac{\partial}{\partial\varphi},h_1(m))\equiv-i\hbar\frac{\partial}{\partial\varphi}-h_1(m),$$

and the wavefunctions are separated:

$$\Psi_{k,l,m}(r,\theta,\varphi)\equiv\langle r,\theta,\varphi|\Psi_{k,l,m}\rangle=R_{k,l}(r)Y_l^m(\theta,\varphi)\ \text{ with }\ Y_l^m(\theta,\varphi)=F_l^m(\theta)e^{im\varphi}.$$

- **How to define the Quantum Separate Variables?**

The main idea introduced by Sklyanin is to use the Yang-Baxter algebra commutation relations to identify a set of quantum separate variables Y_n for the Transfer Matrix $\mathsf{T}(\lambda)$.

- **How to implement SOV systematically for integrable quantum models?**

Still an open problem for some important classes of integrable quantum models. Our research has enlarged the domain of applicability of the SOV approach and allows us to state the next:

- **Motivations to use Quantum Separation of Variables**

SOV method allows to solve the problems which appear in other more traditional methods giving:

a) the proof of completeness of the spectrum description,

b) the analysis of a larger class of integrable quantum models,

c) more symmetrical approach to classical and quantum integrability.

d) universal characterization of spectrum $\&$ dynamics of integrable quantum models.

Sklyanin's approach for the 6-vertex Yang-Baxter case

First observation of integrable structure for Heisenberg spin chain:

Baxter (1972) has first shown that it holds

$$H \propto \frac{d}{d\lambda} \log T(\lambda)_{|\lambda=0,\xi_i=0} + constant$$

between the Hamiltonian $H \in \mathrm{End}(\mathcal{H})$ of the spin 1/2 XYZ quantum chain:

$$H = \sum_{m=1}^{M} \left(J_x \sigma_m^x \sigma_{m+1}^x + J_y \sigma_m^y \sigma_{m+1}^y + J_z \, \sigma_m^z \sigma_{m+1}^z \right) ,$$

quantum space $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \simeq \mathbb{C}^2$, $dim\mathcal{H} = 2^M$, $\sigma_m^{x,y,z}$ Pauli matrices, and the 8-vertex transfer matrix:

$$T(\lambda) = tr_{V_0} R_{0\mathsf{N}}(\lambda - \xi_{\mathsf{N}}) \cdots R_{02}(\lambda - \xi_2) \, R_{01}(\lambda - \xi_1) \in \mathrm{End}(\mathcal{H})$$

and he has shown the commutativity[3] of the transfer matrix:

$$[H, \mathsf{T}(\lambda)] = [\mathsf{T}(\mu), \mathsf{T}(\lambda)] = 0.$$

by the R-matrix properties, named Yang-Baxter equation by Faddeev and his collaborators.

[3] McCoy and Wu (1968) have proven commutativity between the XXZ Hamiltonian and the 6-vertex transfer matrix, while Sutherland (1970) has generalized this for the XYZ Hamiltonian and the 8-vertex transfer matrix.

First observation of integrable structure for Heisenberg spin chain:

The 6-vertex and 8-vertex R-matrices:

$$R_{a,b}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & d(\lambda) \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ d(\lambda) & 0 & 0 & a(\lambda) \end{pmatrix} \in \mathrm{End}(V_a \otimes V_b), \;\; V_i \simeq \mathbb{C}^2$$

is solution of the Yang-Baxter equation[4]

$$R_{ab}(\lambda - \mu) R_{ac}(\lambda) R_{bc}(\mu) = R_{bc}(\mu) R_{ac}(\lambda) R_{ab}(\lambda - \mu) \in \mathbf{End}(V_a \otimes V_b \otimes V_c)$$

- where for the XXX case $a(\lambda) = \lambda + \eta, b(\lambda) = \lambda, c(\lambda) = \eta, d(\lambda) = 0;$

- where for the XXZ case $a(\lambda) = \lambda/q + q/\lambda, b(\lambda) = \lambda + 1/\lambda, c(\lambda) = q + 1/q, d(\lambda) = 0;$

- where for the XYZ case all the coefficients are nonzero elliptic functions.

4 First derived in the context of factorisable scattering processes by Yang (1967).

- **Characterization of integrability by quantum inverse scattering method[5] (QISM)**

– Integrable quantum model with Hamiltonian $H \in \mathrm{End}(\mathcal{H})$ on quantum space $\mathcal{H} \equiv \otimes_{n=1}^{\mathsf{N}} \mathcal{H}_{\mathsf{n}}$:

$$\boxed{\begin{gathered}\underbrace{R_{ab}(\lambda/\mu)\, \mathsf{M}_a(\lambda)\mathsf{M}_b(\mu) = \mathsf{M}_b(\mu)\mathsf{M}_a(\lambda)\, R_{ab}(\lambda/\mu)}_{\text{Yang-Baxter equation}} \in \mathrm{End}(V_a \otimes V_b \otimes \mathcal{H}) \\[2em] \underbrace{\mathsf{M}_a(\lambda) \in \mathrm{End}(V_a \otimes \mathcal{H})}_{\text{Monodromy matrix}},\ \underbrace{\mathsf{T}(\lambda) = \mathrm{tr}_{V_a}\mathsf{M}_a(\lambda)}_{\text{Transfer matrix}},\ \ [H, \mathsf{T}(\lambda)] = [\mathsf{T}(\mu), \mathsf{T}(\lambda)] = 0.\end{gathered}}$$

– If $M_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \in \mathrm{End}(V_a(\simeq \mathbb{C}^2) \otimes \mathcal{H})$, the quantum determinant[6]:

$$\det{}_q \mathsf{M}(\lambda) = A(\lambda)\, D(\lambda/q) - B(\lambda)\, C(\lambda/q)$$

is a central element of the Yang-Baxter algebra associated to the quantum group $U_q(\hat{sl}_2)$.

- **Sklyanin's approach for SOV characterization[7]:** In the 6-vertex case, the quantum separate variables for $\mathsf{T}(\lambda)$ are the operator zeros Y_n of $B(\lambda)$, if $B(\lambda)$ is diagonalizable and with simple spectrum.

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[5] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Teor. Mat. Fiz. 40 (1979) 194.
[6] Izergin and Korepin (1981).
[7] E. K. Sklyanin, Lect. Notes Phys. 226 (1985) 196.

Introduction to SOV & SOV characterization of T-spectrum: "Yang-Baxter case"

- SOV-basis: B-eigenstates $\{\langle \mathbf{y}| \equiv \langle y_1, \ldots, y_\mathsf{N}|\}$ parametrized by the eigenvalues of the zeros.

$$\langle \mathbf{y}|\, B(\lambda) = \, b_\mathbf{y}(\lambda)\, \langle \mathbf{y}|, \;\; b_\mathbf{y}(\lambda) \equiv b_0 \prod_{n=1}^{\mathsf{N}} (\lambda/y_n - y_n/\lambda),$$

The Yang-Baxter commutations relations imply:

$$\langle y_1, \ldots, y_k, \ldots, y_N|A(y_k) = a(y_k)\langle y_1, \ldots, y_k/q, \ldots, y_\mathsf{N}|,$$

$$\langle y_1, \ldots, y_k, \ldots, y_N|D(y_k) = d(y_k)\langle y_1, \ldots, y_k q, \ldots, y_\mathsf{N}|,$$

$$d(\lambda/q)a(\lambda) = \det{}_q M(\lambda).$$

- Eigenvalues $t(\lambda)$ and wavefunctions $\Psi_t(y_1, \ldots., y_\mathsf{N}) \equiv \langle y_1, \ldots, y_\mathsf{N}|t\rangle$ are characterized by:

$$a(y_k)\Psi_t(y_1, .., y_k q^{-1}, .., y_\mathsf{N}) + d(y_k)\Psi_t(y_1, .., y_k q, .., y_\mathsf{N}) {=} t(y_k)\Psi_t(y_1, .., y_\mathsf{N}).$$

They follow by computing the matrix elements $\langle y_1, \ldots., y_\mathsf{N}|\mathsf{T}(y_k)|t\rangle$ and lead to

$$\Psi_t(y_1, \ldots., y_\mathsf{N}) = \prod_{j=1}^{\mathsf{N}} Q_t(y_j),$$

where $Q_t(\lambda)$ is a solution of the Baxter equation.

Derivation of SOV-representations in the Yang-Baxter case

a) Yang-Baxter commutation relation for the 6-vertex R-matrix:

$$A(\mu)B(\lambda)=\frac{q\lambda/\mu-q^{-1}\mu/\lambda}{\lambda/\mu-\mu/\lambda}B(\lambda)A(\mu)+\frac{q^{-1}-q}{\lambda/\mu-\mu/\lambda}B(\mu)A(\lambda)$$

b) The centrality of the quantum determinant:

$$\det_q M(\lambda)\equiv A(\lambda)D(\lambda/q)-B(\lambda)C(\lambda/q)=a(\lambda)d(\lambda/q).$$

- Action of a) for $\mu=y_k$ on the B-eigenstate $\langle y_1,..,y_{\mathsf{N}}|$:

$$(\langle y_1,\dots,y_k,\dots,y_{\mathsf{N}}|A(y_k))\,B(\lambda)=b_{y_1,..,y_k/q,..,y_{\mathsf{N}}}(\lambda)\langle y_1,\dots,y_k,\dots,y_N|A(y_k)$$

where: $b_{y_1,..,y_k/q,..,y_{\mathsf{N}}}(\lambda)\equiv\frac{(q\lambda/y_k-q^{-1}y_k/\lambda)}{(\lambda/y_k-y_k/\lambda)}b_{y_1,..,y_k,..,y_{\mathsf{N}}}(\lambda).$

- Simplicity of $B(\lambda)$ $\rightarrow\langle y_1,\dots,y_k,\dots,y_{\mathsf{N}}|A(y_k)\propto\langle y_1,\dots,y_k/q,\dots,y_{\mathsf{N}}|.$

- Yang-Baxter algebra & quantum determinant imply:

$$\begin{aligned}\langle y_1,\dots,y_k,\dots,y_N|A(y_k)&=a(y_k)\langle y_1,\dots,y_k/q,\dots,y_{\mathsf{N}}|,\\ \langle y_1,\dots,y_k/q,\dots,y_N|D(y_k/q)&=d(y_k/q)\langle y_1,\dots,y_k,\dots,y_{\mathsf{N}}|.\end{aligned}$$

Generalization of Sklyanin's approach for 6-vertex Reflection case

- **Quantum Inverse Scattering Formulation of Open Integrable Quantum Models:**

– Reflection equation[8]:

$$R_{12}(\lambda/\mu)\,\mathcal{U}_{-,1}(\lambda)\,R_{12}(\lambda\mu)\,\mathcal{U}_{-,2}(\mu)=\mathcal{U}_{-,2}(\mu)\,R_{12}(\lambda\mu)\,\mathcal{U}_{-,1}(\lambda)\,R_{12}(\lambda/\mu),$$

– e.g. R-matrix is the 6-vertex one and the boundary monodromy matrix $\mathcal{U}_{-}(\lambda)$ reads[9]:

$$\mathcal{U}_{-,0}(\lambda)=M_0(\lambda)K_{-}(\lambda)\hat{M}_0(\lambda)=\begin{pmatrix}\mathcal{A}_{-}(\lambda) & \mathcal{B}_{-}(\lambda)\\ \mathcal{C}_{-}(\lambda) & \mathcal{D}_{-}(\lambda)\end{pmatrix},\ \hat{M}_0(\lambda)=(-1)^{\mathsf{N}}\,\sigma_0^y\,M_0^{t_0}(1/\lambda)\,\sigma_0^y$$

$M_0(\lambda)$ is a Yang-Baxter equation solution and a scalar solution of the reflection algebra reads[10]

$$K_{\pm}(\lambda)\equiv\begin{pmatrix}a_{\pm}(\lambda) & b_{\pm}(\lambda)\\ c_{\pm}(\lambda) & d_{\pm}(\lambda)\end{pmatrix}\equiv\begin{pmatrix}\lambda\zeta_{\pm}/q^{1/2}-q^{1/2}/(\lambda\zeta_{\pm}) & \kappa_{\pm}e^{\tau_{\pm}}(\lambda^2/q^{\mp1}-q^{\mp1}/\lambda^2)\\ \kappa_{\pm}e^{-\tau_{\pm}}(\lambda^2/q^{\mp1}-q^{\mp1}/\lambda^2) & \zeta_{\pm}q^{1/2}/\lambda-\lambda/(q^{1/2}\zeta_{\pm})\end{pmatrix}$$

– The transfer matrix[9]:

$$\mathsf{T}(\lambda)\equiv\mathrm{tr}_0\{K_{+}(\lambda)\mathcal{U}_{-}(\lambda)\}=a_{+}(\lambda)\,\mathcal{A}_{-}(\lambda)+d_{+}(\lambda)\,\mathcal{D}_{-}(\lambda)+c_{+}(\lambda)\,\mathcal{B}_{-}(\lambda)+b_{+}(\lambda)\,\mathcal{C}_{-}(\lambda)$$

defines a one parameter family of conserved charges for a class of integrable quantum models.

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8 Cherednik I V, 1984 Theor. Math. Phys. 61 977

9 Sklyanin E K, 1988 J. Phys. A: Math. Gen. 21 2375

10 de Vega and Gonz'alez-Ruiz, J. Phys. A 26 (1993) L519.

Introduction to SOV & SOV characterization of T-spectrum: "Reflection algebra case"

- SOV-basis: $\mathcal{B}_-$-eigenstates $\{\langle \mathbf{y}| \equiv \langle y_1, \ldots, y_\mathsf{N}|\}$ parametrized by the eigenvalues of the zeros:

$$\langle \mathbf{y}|\, \mathcal{B}_-(\lambda) = b_{-,\mathbf{y}}(\lambda)\, \langle \mathbf{y}|, \quad b_{-,\mathbf{y}}(\lambda) \equiv b_-(\lambda) \prod_{n=1}^{\mathsf{N}} (\lambda/y_n - y_n/\lambda) \prod_{n=1}^{\mathsf{N}} (\lambda y_n - 1/(y_n\lambda)) \,.$$

The reflection algebra commutations relations imply:

$$\langle y_1, \ldots, y_k, \ldots, y_N|\mathcal{A}_-(y_k^{\pm 1}) = \mathsf{a}_-(y_k^{\pm 1})\langle y_1, \ldots, y_k q^{\mp 1}, \ldots, y_\mathsf{N}|,$$

$$\mathcal{D}_-(\lambda) = \left[(\lambda^2/q - q/\lambda^2)\mathcal{A}_-(1/\lambda) + (\lambda^2/q - q/\lambda^2)\mathcal{A}_-(\lambda)\right]/(\lambda^2 - 1/\lambda^2),$$

$$\mathrm{det}_q\,\mathcal{U}_-(\lambda)/(\lambda^2/q - q/\lambda^2) = \mathcal{A}_-(\lambda q^{1/2})\mathcal{A}_-(q^{1/2}/\lambda) + \mathcal{B}_-(\lambda q^{1/2})\mathcal{C}_-(q^{1/2}/\lambda) = \mathsf{a}_-(\lambda q^{1/2})\mathsf{a}_-(q^{1/2}/\lambda).$$

- In the triangular case ($b_+(\lambda) = 0$) the transfer matrix reads:

$$\mathsf{T}(\lambda) \equiv \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\} = \bar{a}_+(\lambda)\,\mathcal{A}_-(\lambda) + \bar{a}_+(1/\lambda)\,\mathcal{A}_-(1/\lambda) + c_+(\lambda)\,\mathcal{B}_-(\lambda).$$

defined $\mathsf{a}(\lambda) \equiv \bar{a}_+(\lambda)\mathsf{a}_-(\lambda)$ then the eigenvalues and eigenstates are characterized by:

$$\mathsf{a}(y_k)\Psi_t(y_1, .., y_k q^{-1}, .., y_\mathsf{N}) + \mathsf{a}(1/y_k)\Psi_t(y_1, .., y_k q, .., y_\mathsf{N}) = t(y_k)\Psi_t(y_1, .., y_\mathsf{N}).$$

These equations lead to factorized wavefunctions by Baxter equation solutions $Q_t(y_j)$:

$$\Psi_t(y_1, \ldots., y_\mathsf{N}) = \prod_{j=1}^{N} Q_t(y_j).$$

General results on the spectrum

Results in the SOV framework:

• Systematic SOV development and new exact results for integrable quantum models like:

◦ The lattice sine-Gordon model, the chiral Potts model and their generalizations associated to general cyclic representations of the Yang-Baxter and Reflection Algebras.

◦ The XYZ quantum spin 1/2 chains and XXZ with arbitrary spin representations and the most general integrable closed/open boundary conditions.

◦ The SOS models associated to spin 1/2 representations of the dynamical Yang-Baxter and Reflection Algebras of elliptic and trigonometric type.

• Universal features proven for integrable quantum models associated to representations of the Yang-Baxter and Reflection Algebras of $U_q(\hat{sl}(2))$ and $\mathsf{E}_{\tau,q}(\hat{sl}(2))$ type:

Spectrum: the eigenvalues and eigenstates of the Hamiltonian of the model are completely characterized by classifying all the solutions to a given set of equations (Baxter's second order difference equations) in the spectrum of the separate variables:

$$\mathbf{T}(\lambda)\,|t\rangle = t(\lambda)|t\rangle, \quad |t\rangle \text{ **eigenvector of** } \mathbf{T}(\lambda), \; t(\lambda) \text{ **eigenvalue of** } \mathbf{T}(\lambda),$$

$$\Updownarrow$$

SOV representation: $|t\rangle = \sum_{\{y\}} \prod_{j=1}^{N} Q_t(y_j)|y_1, \ldots, y_N\rangle, \qquad Q_t(y_j) \in \mathbb{C},$

Baxter's equation: $t(y_j)Q_t(y_j) = a(y_j)Q_t(y_j/q) + d(y_j)Q_t(y_j q), \quad q \in \mathbb{C}.$

Open chain associated to cyclic representations[11]

[11] Maillet, Niccoli, Pezelier, Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra I, SciPost Phys. 2, 009 (2017);
——— II: the most general integrable boundary conditions arXiv:1802.08853;
——— III: locals Hamiltonians to appear.

The τ_2-model: QISM framework

The Lax operator[12]:

$$\mathbf{L}_{0,n}(\lambda) \equiv \left(\begin{array}{cc} \lambda\alpha_n V_n - \beta_n \lambda^{-1} V_n^{-1} & U_n \left(q^{-1/2} \mathbb{a}_n V_n + q^{1/2} \mathbb{b}_n V_n^{-1} \right) \\ U_n^{-1} \left(q^{1/2} \mathbb{c}_n V_n + q^{-1/2} \mathbb{d}_n V_n^{-1} \right) & \gamma_n V_n / \lambda - \delta_n \lambda / V_n \end{array} \right)_0$$

$U_n V_m = q^{\delta_{n,m}} V_m U_n$, $\forall n, m \in \{1, ..., \mathsf{N}\}$, $q = e^{-2i\pi p'/p}$, p odd and p', p integer coprime, α_n, β_n, γ_n, δ_n, $\mathbb{a}_n$, $\mathbb{b}_n$, $\mathbb{c}_n$ and $\mathbb{d}_n$ are parameters satisfying $\alpha_n \gamma_n = \mathbb{a}_n \mathbb{c}_n$, $\beta_n \delta_n = \mathbb{b}_n \mathbb{d}_n$.

Cyclic representations of local Weyl algebras: U_n and V_n are unitary operators and U_n^p, V_n^p are centrals, we fix $U_n^p = V_n^p = 1$, and then we can introduced the following p-dimensional representations for each local Weyl algebra:

$$V_n |k, n\rangle = q^k |k, n\rangle \quad U_n |k, n\rangle = |k + 1, n\rangle \quad \forall k \in \{1, ..., p\}$$

by the action on an eigenbasis of V_n with the cyclicity conditions: $|k + p, n\rangle = |k, n\rangle$.

[12] Bazhanov and Stroganov, J. Stat. Phys. 59 (1990) 799; Baxter, Bazhanov and Perk Int. J. Mod. Phys. B4 (1990) 803, see von Gehlen et al 2006-08 some first results in SOV for the periodic chain. For special values of the parameters this Lax operator reduces to the one first introduced by Izerging and Korepin, Nuclear Physics B205 [FS5] (1982) 401.

Construction of SOV basis: By using Baxter's like gauge transformations we get

$$\mathcal{A}_-(\lambda|\beta) = [-(\lambda q^{3/2}/\beta)\mathcal{A}_-(\lambda) - \alpha q\mathcal{B}_-(\lambda) + \mathcal{C}_-(\lambda)/(\alpha q) + \beta\mathcal{D}_-(\lambda)/(\lambda q^{3/2})]/(\beta/q^2 - q^2/\beta)$$

$$\mathcal{B}_-(\lambda|\beta) = [-(\lambda\beta/q^{1/2})\mathcal{A}_-(\lambda) - \alpha q\mathcal{B}_-(\lambda) + (\beta^2/\alpha q)\mathcal{C}_-(\lambda) + (\beta q^{1/2}/\lambda)\mathcal{D}_-(\lambda)]/(\beta - 1/\beta)$$

$$\mathcal{D}_-(\lambda|\beta) = [(\lambda\beta/q^{1/2})\mathcal{A}_-(\lambda) + \alpha q\mathcal{B}_-(\lambda) - \mathcal{C}_-(\lambda)/\alpha q - (q^{1/2}/\lambda\beta)\mathcal{D}_-(\lambda)]/(\beta - 1/\beta),$$

where $\beta \neq \pm 1, \pm q^2$, for a special choice of the free gauge parameter α, it holds

$$\mathcal{T}(\lambda) = \bar{a}_+(\lambda)\mathcal{A}_-(\lambda|\beta q^2) + \bar{a}_+(1/\lambda)\mathcal{A}_-(1/\lambda|\beta q^2) + q\bar{c}_+(\lambda|\beta)\mathcal{B}_-(\lambda|\beta)$$
$$\mathcal{T}(\lambda) = \bar{d}_+(\lambda)\mathcal{D}_-(\lambda|\beta) + \bar{d}_+(1/\lambda)\mathcal{D}_-(1/\lambda|\beta) + \bar{c}_+(\lambda|\beta)\mathcal{B}_-(\lambda|\beta)/q,$$

Proposition $\mathcal{B}_-(\lambda|\beta)$ is pseudo diagonalizable and non-degenerate:
Maillet, Niccoli, Pezelier (2018)

$$\langle\beta,\mathbf{h}|\mathcal{B}_-(\lambda|\beta) = \mathrm{B}_{\mathbf{h}}(\lambda|\beta)\langle\beta/q^2,\mathbf{h}|, \quad \mathcal{B}_-(\lambda|\beta)|\beta,\mathbf{h}\rangle = |q^2\beta,\mathbf{h}\rangle\mathrm{B}_{\mathbf{h}}(\lambda|\beta),$$
$$\langle\beta, h_1, ..., h_n, ..., h_{\mathsf{N}}|\mathcal{A}_-((y_a^{(h_a)})^{\pm 1}|\beta q^2) = \mathsf{a}_-((y_a^{(h_a)})^{\pm 1})\langle\beta, h_1, ..., h_n \mp 1, ..., h_{\mathsf{N}}|$$

where $\mathbf{h} \equiv (h_1, ..., h_{\mathsf{N}})$, $y_n^{(h)} = y_n^{(0)}q^h$, $\quad h \in \{0, ..., p-1\}$, $n \in \{1, ..., 2\mathsf{N}\}$,

$$\mathrm{B}_{\mathbf{h}}(\lambda|\beta) = \mathrm{B}_-(\beta)(\frac{\lambda^2}{q} - \frac{q}{\lambda^2})\prod_{a=1}^{\mathsf{N}}(\frac{\lambda}{y_a^{(h_a)}} - \frac{y_a^{(h_a)}}{\lambda})(\lambda y_a^{(h_a)} - \frac{1}{\lambda y_a^{(h_a)}}).$$

SOV characterization of the spectrum: We define $\mathsf{a}(\lambda) = \bar{a}_{+}(\lambda)\mathsf{a}_{-}(\lambda)$ and

$$D_{\mathcal{T}}(\lambda) \equiv \begin{pmatrix} \tau(\lambda) & -\mathsf{a}(1/\lambda) & 0 & \cdots & 0 & -\mathsf{a}(\lambda) \\ -\mathsf{a}(q\lambda) & \tau(q\lambda) & -\mathsf{a}(1/(q\lambda)) & 0 & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & \cdots & & & \vdots \\ \vdots & & & \cdots & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & 0 & -\mathsf{a}(q^{2l-1}\lambda) & \tau(q^{2l-1}\lambda) & -\mathsf{a}(1/\left(q^{2l-1}\lambda\right)) \\ -\mathsf{a}(1/\left(q^{2l}\lambda\right)) & 0 & \cdots & 0 & -\mathsf{a}(q^{2l}\lambda) & \tau(q^{2l}\lambda) \end{pmatrix},$$

Theorem 1 $\mathcal{T}(\lambda)$ is diagonalizable and has non-degenerate spectrum $\Sigma_{\mathcal{T}}$ characterized by:
Maillet, Niccoli, Pezelier (2018)

$$\det D_{\tau}(y_a^{(0)}) = 0, \ \forall a \in \{1, \dots, \mathsf{N}\},$$

where $\tau(\lambda)$ are polynomial of degree N+2 in $\Lambda \equiv (\lambda^2 + 1/\lambda^2)$, $\mathcal{T}(\lambda)$ has knwon central asymptotic τ_∞ and knwon central values in $\pm q$, $\pm i\, q$.
The associated eigenvector/eigencovector are uniquely fixed, up to a normalization, by:

$$\langle h_1, \dots, h_{\mathsf{N}}, \beta/q^2|\tau\rangle = \prod_{a=1}^{\mathsf{N}} q_{\tau,a}^{(h_a)}, \quad \langle\tau|\beta, h_1, \dots, h_{\mathsf{N}}\rangle = \prod_{a=1}^{\mathsf{N}} \hat{q}_{\tau,a}^{(h_a)}, \quad \forall\tau(\lambda) \in \Sigma_{\mathcal{T}}$$

where the $q_{\tau,a}^{(h_a)}$ are the unique nontrivial solutions up to normalization of the linear homogeneous system (similar characterization for $\hat{q}_{\tau,a}^{(h_a)}$):

$$\left(q_{\tau,a}^{(0)}, \dots, q_{\tau,a}^{(p-1)}\right)\left(D_{\tau}(y_a^{(0)})\right)^{t} = (0, \dots, 0)\,.$$

Theorem 2 Maillet, Niccoli, Pezelier (2018) $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ if and only if $\tau(\lambda)$ is an entire function and there exists a unique polynomial

$$Q(\lambda) = \prod_{b=1}^{\mathsf{N}_Q} (\Lambda - \Lambda_b)$$

satisfying the following functional equation:

$$\tau(\lambda)Q(\lambda) = \mathsf{a}(\lambda)Q(\lambda/q) + \mathsf{a}(1/\lambda)Q(\lambda q) + \left[\tau_\infty - (\mathsf{a}_\infty q^{-\mathsf{N}_Q} + \mathsf{a}_0 q^{\mathsf{N}_Q})\right](\Lambda^2 - X^2)F(\lambda),$$

and the conditions:

$$(Q(y_a^{(0)}), \ldots, Q(y_a^{(p-1)})) \neq (0, \ldots, 0) \quad \forall a \in \{1, \ldots, \mathsf{N}\}.$$

Here, $\mathsf{N}_Q = (p-1)\,\mathsf{N}$ for inhomogeneous, $\mathsf{N}_Q \leq (p-1)\,\mathsf{N}$ for homogeneous equation and

$$F(\lambda) = \prod_{b=1}^{2\mathsf{N}} \left(\frac{\lambda^p}{(y_b^{(0)})^p} - \frac{(y_b^{(0)})^p}{\lambda^p}\right),$$

the associated eigenvector/eigencovector reads, up to an overall normalization:

$$\langle h_1, \ldots, h_{\mathsf{N}}, \beta/q^2 | \tau\rangle = \prod_{a=1}^{\mathsf{N}} Q(y_a^{(h_a)}), \quad \forall \tau(\lambda) \in \Sigma_{\mathcal{T}}.$$

General results toward quantum dynamics[13]

[13] Kitanine, Maillet, Niccoli, Terras, The open XXX spin chain in the SoV framework: scalar product of separate states, (2017) J. Phys. A: Math. Theor. 50 224001,
——— The open XXZ spin chain in the SoV framework: scalar product of separate states, to appear.

Recall of some known result known results for XXZ chain dynamics

- Free fermion point, representations as solution of Painlevé equations:

  $\hookrightarrow$ **1961 Lieb, Shultz, Mattis, since 1976 Wu, McCoy, Tracy, Barouch, Sato, Jimbo, Miwa, Zamolodchikov**

- First analysis by using Bethe ansatz for general anisotropy Δ since 1984 : Izergin, Korepin,...

- Interacting case $\Delta \neq 0$, multiple integral representations:

  - Q-deformed KZ equations: (Zero External Magnetic Field)

    $\hookrightarrow$ **Infinite and half-infinite Chains: 1992-1996 Jimbo, Kedem, Kojima, Konno, Miki, Miwa, Nakayashiki**

  - In the framework of algebraic Bethe ansatz: (Zero&Non-Zero External Magnetic Field)

    $\hookrightarrow$ **Periodic Chain, *Lyon Group*: 1999 Kitanine, Maillet, Terras**

    $\hookrightarrow$ **Open diagonal chain, *Lyon Group*: 2007 Kitanine, Kozlowski, Maillet, Niccoli, Slavnov, Terras.**

- Interacting case $\Delta \neq 0$, measurable quantities and asymptotic behavior:

$\hookrightarrow$ **Periodic XXZ chain: Dynamical Structure Functions. since 2005 Caux and Maillet.**

$\hookrightarrow$ **Periodic XXZ chain (and 1D Bose Gas): Asymptotic Behavior of Two-Point Functions for Non-Zero Magnetic Field. *Lyon Group*: since 2009 Kitanine, Kozlowski, Maillet, Slavnov, Terras.**

Several other developments on the quantum dynamics, e.g. the temperature case and higher rank case: Boos, Gohmann, Jimbo, Klumper, Kozlowski, Lukyanov, Miwa, Pakuliak, Ragoucy, Ribeiro, Slavnov, Seel, Smirnov, Suzuki, Takeyama ...

Definition and problems to solve:

Definition Form factors $\langle t'|\mathcal{O}_n|t\rangle$ are the matrix elements of a local operator $\mathcal{O}_n$ between the eigencovector $\langle t'|$ and the eigenvector $|t\rangle$ of $\mathsf{T}(\lambda)$.

The form factors are the "elementary objects" w.r.t. any time dependent correlation function can be expanded by using the decomposition of the identity in the transfer matrix eigenbasis:

$$\langle t'|\mathcal{O}_n(\theta_1)\mathcal{O}_m(\theta_2)|t''\rangle = \sum_{t\in\Sigma_{\mathsf{T}}} \frac{\langle t'|\mathcal{O}_n|t\rangle\langle t|\mathcal{O}_m|t''\rangle}{\langle t|t\rangle} e^{(h_{t'}-h_t)\theta_1+(h_t-h_{t''})\theta_2}, \forall n<m \in \{1,\ldots,\mathsf{N}\}$$

where $h_{t'}$ and $h_{t''}$ are the Hamiltonian eigenvalues on the eigenstates $|t''\rangle$ and $|t'\rangle$ and by definition of time evolution operator, it holds $\mathcal{O}_n(\theta) \equiv e^{iH\theta}\mathcal{O}_n e^{-iH\theta}$.

Two difficult problems to solve:

i) Reconstruction of local operators $\mathcal{O}_n$ in terms of quantum SOV variables and their conjugates.

$\hookrightarrow$ Algebraic computation of the action of local operators $\mathcal{O}$ on the eigenvector $|t\rangle$.

ii) Scalar product $\langle\alpha|t\rangle$ under the form of determinant, where $\langle\alpha|$ is a generic separate state.

Steps i) and ii) allow us to get in a determinant form the form factors of a basis of operators.

First general results by SOV:

Universal characterization of the form factors by SOV-method[14]:

For integrable models associated to finite dimensional quantum space, there exists a basis $\mathbb{B}_{\mathcal{H}}$ in End$(\mathcal{H})$ such that for any $\mathsf{O} \in \mathbb{B}_{\mathcal{H}}$ the matrix elements read:

$$\langle t'|\mathsf{O}|t\rangle = \det_{\mathsf{N}} ||\Phi_{a,b}^{(\mathsf{O},t,t)}||, \quad \Phi_{a,b}^{(\mathsf{O},t,t)} \equiv \sum_{c=1}^{p} F_{\mathsf{O},b}(y_a^{(c)}) Q_t(y_a^{(c)}) Q_{t'}(y_a^{(c)}) (y_a^{(c)})^{2(b-1)}.$$

$F_{\mathsf{O},b}()$ characterize the operator O and are computed algebraically by SOV reconstruction of O.

For XXX/XXZ spin 1/2 chains with closed or open integrable boundaries[15] scalar products and form factors admit rewriting by Izergin's and Slavnov's type formulae[16].

Important achievement to apply known techniques for the thermodynamic limit analysis to integrable quantum models previously not analyzable.

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[14] Grosjean, Maillet, Niccoli (2012) and subsequent papers.

[15] Kitanine, Maillet, Niccoli, Terras (2015-2018)

[16] For closed XXX chain they are reminiscent of ones first obtained by Kostov (2012) by another approach.

The explicit example of the open XXX quantum spin-1/2 chain

The XXX quantum spin-1/2 chain with the most general integrable boundaries

- The local Hamiltonian, the rational 6-vertex monodromy matrix and R-matrix:

$$H=\sum_{i=1}^{N-1}\left[\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y+\sigma_i^z\sigma_{i+1}^z\right]+\frac{\eta}{\zeta_-}\left[\sigma_1^z+2\kappa_-\left(e^{\tau_-}\sigma_1^++e^{-\tau_-}\sigma_1^-\right)\right]+\frac{\eta}{\zeta_+}\left[\sigma_N^z+2\kappa_+\left(e^{\tau_+}\sigma_N^++e^{-\tau_+}\sigma_N^-\right)\right]$$

- The transfer matrix $\mathcal{T}(\lambda)\equiv\mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\}$, associated to the triangular boundaries

$$K_+(\lambda)=\mathrm{I}+\frac{\lambda+\eta/2}{\bar{\zeta}_+}(\sigma^z+\bar{\mathsf{c}}_+\sigma^-),\qquad K_-(\lambda)=\mathrm{I}+\frac{\lambda-\eta/2}{\bar{\zeta}_-}(\sigma^z+\bar{\mathsf{b}}_-\sigma^+),$$

with $\bar{\zeta}_\pm$, $\bar{\mathsf{c}}_+$, $\bar{\mathsf{b}}_-$ fixed by the original boundaries parameters by $\bar{\zeta}_\pm=\zeta_\pm/\sqrt{1+4\kappa_\pm^2}$,

$$\bar{\mathsf{c}}_+=\frac{2\kappa_+e^{-\tau_+}\left[1+\dfrac{(1+\sqrt{1+4\kappa_+^2})(1-\sqrt{1+4\kappa_-^2})}{4\kappa_+\kappa_-e^{\tau_--\tau_+}}\right]}{\sqrt{1+4\kappa_+^2}},\quad \bar{\mathsf{b}}_-=\frac{2\kappa_-e^{\tau_-}\left[1+\dfrac{(1-\sqrt{1+4\kappa_+^2})(1+\sqrt{1+4\kappa_-^2})}{4\kappa_+\kappa_-e^{\tau_--\tau_+}}\right]}{\sqrt{1+4\kappa_-^2}},$$

defines this local Hamiltonian by the similarity matrix $\Gamma_W\equiv\otimes_{n=1}^N W_n$ as it follows :

$$H=\frac{1}{2\,\eta^{2N-1}}\Gamma_W^{-1}\frac{d}{d\lambda}\mathcal{T}(\lambda)_{|\lambda=\eta/2,\xi_i=0}\Gamma_W\,,\quad W_n\equiv\mathrm{I}_n-\frac{1-\sqrt{1+4\kappa_+^2}}{2\kappa_+e^{-\tau_+}}\sigma_n^++\frac{1-\sqrt{1+4\kappa_-^2}}{2\kappa_-e^{\tau_-}}\sigma_n^-.$$

SOV characterization of the transfer matrix spectrum by discrete system of equations

Theorem 1. i) *SOV basis exists for generic inhomogeneities and* $\bar{\mathsf{b}}_- \neq 0$*:*

$$\begin{aligned}\langle \mathbf{h}|\mathcal{B}_-(\lambda) &= \mathsf{b}_{-,\mathbf{h}}(\lambda)\langle \mathbf{h}|,\\ \mathcal{B}_-(\lambda)|\mathbf{h}\rangle &= \mathsf{b}_{-,\mathbf{h}}(\lambda)|\mathbf{h}\rangle,\end{aligned} \quad \text{where} \quad \mathsf{b}_{-,\mathbf{h}}(\lambda) = \bar{\mathsf{b}}_- \frac{\lambda-\eta/2}{\bar{\zeta}_-}\prod_{n=1}^{N}(\lambda^2 - (\xi_n^{(h_n)})^2),$$

$$\langle \mathbf{k}|\mathbf{h}\rangle = \delta_{\mathbf{h},\mathbf{k}}\, N_{\boldsymbol{\xi},-}/\widehat{V}(\{\xi_r^{(h_r)}\}), \widehat{V}(\{x_h\}) = \prod_{j<k}(x_k^2 - x_j^2), \xi_r^{(h)} = \xi_r + \eta/2 - h\eta.$$

ii) *The spectrum* $\Sigma_{\mathcal{T}}$ *of* $\mathcal{T}(\lambda)$ *is simple and defined by*

$$t(\xi_n^{(0)})\, t(\xi_n^{(1)}) = \mathsf{a}(\xi_n^{(0)})\, \mathsf{a}(-\xi_n^{(1)}),\ \ 1 \leqslant n \leqslant N,$$

with $t(\lambda) = \frac{2+\bar{\mathsf{b}}_-\bar{\mathsf{c}}_+}{\bar{\zeta}_+\bar{\zeta}_-}(\lambda^2 - (\eta/2)^2)\prod_{b=1}^{N}(\lambda^2 - t_b^2) + 2(-1)^N a(0)\, d(-\eta)$*:*

$$\mathsf{a}(\lambda) \equiv (-1)^N \frac{2\lambda+\eta}{2\lambda}\frac{(\lambda - \frac{\eta}{2} + \bar{\zeta}_+)(\lambda - \frac{\eta}{2} + \bar{\zeta}_-)a(\lambda)\, d(-\lambda)}{\bar{\zeta}_+\bar{\zeta}_-},\ d(\lambda+\eta) \equiv a(\lambda) \equiv \prod_{n=1}^{N}(\lambda - \xi_n + \eta/2).$$

iii) *The right and left* $\mathcal{T}(\lambda)$*-eigenstates associated to* $t(\lambda)$ *read:*

$$\langle \mathbf{h}|t\rangle = \prod_{n=1}^{N} Q_t(\xi_n^{(h_n)}),\ \ \langle t|\mathbf{h}\rangle = \prod_{n=1}^{N}(f_n\, g_n)^{h_n}\, Q_t(\xi_n^{(h_n)}),\quad \frac{Q_t(\xi_n^{(1)})}{Q_t(\xi_n^{(0)})} = \frac{t(\xi_n^{(0)})}{\mathsf{a}(\xi_n^{(0)})},$$

where $g_n \equiv (\xi_n + \bar{\zeta}_+)(\xi_n + \bar{\zeta}_-)/((\xi_n - \bar{\zeta}_+)(\xi_n - \bar{\zeta}_-))$, *and* $f_n \equiv \prod_{a=1\, a\neq n}^{N}\frac{(\xi_n - \xi_a + \eta)(\xi_n + \xi_a + \eta)}{(\xi_a - \xi_n + \eta)(\xi_a + \xi_n - \eta)}$.

SOV characterization of the transfer matrix spectrum by TQ functional equations

Theorem 2. a) $t(\lambda) \in \Sigma_{\mathcal{T}}$ *iff* $\exists!\ Q_t(\lambda) = \prod_{b=1}^{q} \left(\lambda^2 - \lambda_b^2\right)$ *with* $\lambda_i \neq \xi_j^{(0)}\ \forall i, j$:

$$t(\lambda)\, Q_t(\lambda) = \mathsf{a}(\lambda)\, Q_t(\lambda - \eta) + \mathsf{a}(-\lambda)\, Q_t(\lambda + \eta) + F(\lambda),$$

$F(\lambda) = \frac{\bar{\mathsf{b}}_- \bar{\mathsf{c}}_+}{\bar{\zeta}_- \bar{\zeta}_+}(\lambda^2 - (\eta/2)^2) \prod_{b=1}^{N} \prod_{h=0}^{1} (\lambda^2 - (\xi_b^{(h)})^2)$, $q{=}N$ *for* $\bar{\mathsf{c}}_+ \neq 0$ *and* $q \leq N$ *for* $\bar{\mathsf{c}}_+ = 0$.

b) *Algebraic Bethe ansatz form of transfer matrix eigenstates:*

$$|t\rangle = \prod_{a=1}^{q} \mathcal{B}(\lambda_a)|\omega_R\rangle \text{ and } \langle t| = \langle \omega_L| \prod_{a=1}^{q} \mathcal{B}(\lambda_a), \text{ where :}$$

$$\mathcal{B}(\lambda) \equiv \frac{(-1)^N \bar{\zeta}_- \mathcal{B}_-(\lambda)}{(\lambda - \eta/2)\bar{\mathsf{b}}_-},\ |\omega_R\rangle = \sum_{\mathbf{h}\in\{0,1\}^N} \frac{\widehat{V}(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)})|\mathbf{h}\rangle}{N_{\boldsymbol{\xi},-}},\ \langle \omega_L| = \sum_{\mathbf{h}\in\{0,1\}^N} \frac{\widehat{V}(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)})\langle \mathbf{h}|}{N_{\boldsymbol{\xi},-} \prod_{a=1}^{N}(f_a\, g_a)^{-h_a}}.$$

Def: $|\alpha\rangle$, $\langle\beta|$ are separate states iff $\langle \mathbf{h}|\alpha\rangle = \prod_{n=1}^{N} \alpha_n^{(h_n)}$, $\langle \beta|\mathbf{h}\,\rangle = \prod_{n=1}^{N} (f_n\, g_n)^{h_n} \beta_n^{(h_n)}$.

Remark:
$$|\alpha\rangle = \prod_{a=1}^{n_\alpha} \mathcal{B}(\alpha_a)|\omega_R\rangle, \quad \langle\beta| = \langle \omega_L| \prod_{a=1}^{n_\beta} \mathcal{B}(\beta_a)$$

if $\alpha(\lambda) = \prod_{k=1}^{n_\alpha}(\lambda^2 - \alpha_k^2)$, $\beta(\lambda) = \prod_{k=1}^{n_\beta}(\lambda^2 - \beta_k^2)$: $\alpha(\xi_n^{(h_n)}) = \alpha_n^{(h_n)}$, $\beta(\xi_n^{(h_n)}) = \beta_n^{(h_n)}$.

Scalar product of separate states: first SOV representation

For a set of arbitrary variables $\{x\} \equiv \{x_1, \ldots, x_L\}$ and a function f, we define the function

$$\mathcal{A}_{\{x\}}[f] = \det_{1\leqslant i,j\leqslant L}\left[\sum_{\epsilon\in\{+,-\}} f(\epsilon x_i)\left(x_i + \epsilon\frac{\eta}{2}\right)^{2(j-1)}\right]\det^{-1}_{1\leqslant i,j\leqslant L}\left[x_i^{2(j-1)}\right],$$

and

$$f_{\xi_+,\xi_-,\{z\}}(\lambda) = \lambda^{-1}(\lambda+\xi_+)(\lambda+\xi_-)\prod_{m=1}^{M}(\lambda^2 - z_m^2)/((\lambda+\frac{\eta}{2})^2 - z_m^2),$$

for ξ_+, ξ_- and $\{z\} \equiv \{z_1, \ldots, z_M\}$ a set of arbitrary variables.

Proposition 1. *The scalar products of the separate states*

$$\langle \alpha \mid \beta\rangle = \langle \beta \mid \alpha\rangle = \langle \omega_L|\prod_{k=1}^{n_\alpha}\mathcal{B}(\alpha_k)\prod_{k=1}^{n_\beta}\mathcal{B}(\beta_k)|\omega_R\rangle,$$

admit the following determinant representations:

$$\langle \alpha \mid \beta\rangle = N_{\alpha,\beta,\bar{\zeta}_+}\,\mathcal{A}_{\{\xi\}}\left[f_{\bar{\zeta}_+,\bar{\zeta}_-,\{\alpha\}\cup\{\beta\}}\right],$$

where we have defined

$$N_{\alpha,\beta,\bar{\zeta}_+} \equiv (-1)^N\prod_{n=1}^{N}\frac{\alpha(\xi_n^{(0)})\,\beta(\xi_n^{(0)})\,\alpha(\xi_n^{(1)})\,\beta(\xi_n^{(1)})}{(\xi_n - \bar{\zeta}_+)\,\bar{\mathsf{b}}_-\,\alpha(\xi_n)\,\beta(\xi_n)}.$$

Main determinant identities to rewrite scalar product of separate states

Identity 1. *Let* $\{x\} \equiv \{x_1, \ldots, x_L\}$ *and* $\{z\} \equiv \{z_1, \ldots, z_M\}$ *be two sets of arbitrary complex numbers and let* $\theta(x) \equiv \{0 \text{ if } x < 0, 1 \text{ if } x \geq 0\}$. *Then, if* ξ_+ *and* ξ_- *are such that* $(\xi_+ + \xi_-)/\eta \notin \{1, \ldots, M - L\}$ *(non-empty set only for* $M > L$*), it holds:*

$$\mathcal{A}_{\{x\}}\left[f_{\xi_+,\xi_-,\{z\}}\right] = p_{\xi_+ + \xi_-,L,M}\ \mathcal{A}_{\{z\}}\left[f_{\frac{\eta}{2}-\xi_+,\frac{\eta}{2}-\xi_-,\{x\}}\right]$$

where $p_{x,L,M} \equiv (-1)^M \prod_{a=0}^{|L-M|-1}(2(x + \eta(a + (L-M)\theta(M-L))))^{1-2\theta(M-L)}$.

For $L \geqslant M$ we define the function $\mathcal{S}_{\{x\},\{y\}}[f] = \frac{\widehat{V}(x_1-\frac{\eta}{2},\ldots,x_M-\frac{\eta}{2})}{\widehat{V}(x_1+\frac{\eta}{2},\ldots,x_M+\frac{\eta}{2})} \frac{\det_L \mathcal{S}_{\mathbf{x},\mathbf{y}}[f]}{\widehat{V}(x_M,\ldots,x_1)\,\widehat{V}(y_1,\ldots,y_L)}$,

$$\left[\,\mathcal{S}_{\mathbf{x},\mathbf{y}}[f]\,\right]_{i,k} = \sum_{\epsilon\in\{+,-\}} f(\epsilon y_i) X(y_i+\epsilon\eta) \begin{cases} \frac{f(-x_k)}{(y_i+\epsilon\frac{\eta}{2})^2-(x_k+\frac{\eta}{2})^2} - \frac{f(x_k)\varphi_{\{x\}}(x_k)}{(y_i+\epsilon\frac{\eta}{2})^2-(x_k-\frac{\eta}{2})^2} & \text{if } k \leqslant M, \\ \left(y_i+\epsilon\frac{\eta}{2}\right)^{2(k-M-1)} & \text{if } k > M, \end{cases}$$

$\varphi_{\{x\}}(\lambda) = (2\lambda - \eta)X(\lambda+\eta)/((2\lambda+\eta)X(\lambda-\eta))$ and $X(\lambda) = \prod_{m=1}^{M}(\lambda^2 - x_m^2)$.

Identity 2. *Let us suppose that* $L \geqslant M$, *then for any function* f,

$$\mathcal{A}_{\{x\}\cup\{y\}}[f] = \mathcal{S}_{\{x\},\{y\}}[f].$$

Scalar product of separate states: new SOV representation

Theorem 3. *Let $n_\beta \geqslant n_\alpha$, then the scalar product of the separate states $\langle\alpha|$ and $|\beta\rangle$ reads:*

$$\langle\alpha|\beta\rangle = N_{\alpha,\beta,\bar{\zeta}_+}\, p_{\bar{\zeta}_+ + \bar{\zeta}_-,N,n_\alpha+n_\beta}\, \mathcal{A}_{\{\alpha\}\cup\{\beta\}}\left[f_{-\bar{\zeta}_+ + \frac{\eta}{2},-\bar{\zeta}_- + \frac{\eta}{2},\{\xi\}}\right]$$

or:

$$\langle\alpha|\beta\rangle = N_{\alpha,\beta,\bar{\zeta}_+}\, p_{\bar{\zeta}_+ + \bar{\zeta}_-,N,n_\alpha+n_\beta}\, \mathcal{S}_{\{\alpha\},\{\beta\}}\left[f_{-\bar{\zeta}_+ + \frac{\eta}{2},-\bar{\zeta}_- + \frac{\eta}{2},\{\xi\}}\right].$$

Let $\bar{\mathsf{c}}_+ = 0$, then the scalar product $\langle t|\beta\rangle$=$\langle\beta|t\rangle$=0 if $n_\beta < q$, while for $n_\beta \geqslant q$ it reads:

$$\langle \mathrm{t}|\beta\rangle = \frac{(-1)^{q+n_\beta} p_{\bar{\zeta}_+ + \bar{\zeta}_-,N,q+n_\beta}}{\prod_{n=1}^{N}\left[(\xi_n - \bar{\zeta}_+)\,\bar{\mathsf{b}}_-\right]} \prod_{k=1}^{q} \frac{(\lambda_k - \frac{\eta}{2} + \bar{\zeta}_+)(\lambda_k - \frac{\eta}{2} + \bar{\zeta}_-)}{\lambda_k}\, a(\lambda_k)\, d(-\lambda_k)$$

$$\times \prod_{i=1}^{n_\beta} \frac{2\bar{\zeta}_+\bar{\zeta}_-\, Q_t(\beta_i)}{\eta^2 - 4\beta_i^2}\, \frac{\widehat{V}(\lambda_1 - \frac{\eta}{2},\ldots,\lambda_q - \frac{\eta}{2})}{\widehat{V}(\lambda_1 + \frac{\eta}{2},\ldots,\lambda_q + \frac{\eta}{2})}\, \frac{\det_{\mathbf{n}_\beta} \mathcal{S}_{\mathbf{t}}(\{\beta\})}{\widehat{V}(\lambda_q,\ldots,\lambda_1)\,\widehat{V}(\beta_1,\ldots,\beta_{n_\beta})},$$

where

$$\left[\,\mathcal{S}_t(\{\beta\})\right]_{i,k} = \begin{cases} \dfrac{\partial\, t(\beta_i)}{\partial \lambda_k} & \textbf{if } k \leqslant q, \\[2ex] \displaystyle\sum_{\epsilon\in\{+,-\}} \epsilon\, \mathsf{a}(-\epsilon\beta_i)\, \frac{Q_t(\beta_i + \epsilon\eta)}{Q_t(\beta_i)} \left(\beta_i + \epsilon\frac{\eta}{2}\right)^{2(k-q)-1} & \textbf{if } k > q. \end{cases}$$

Related projects: I am developing with my collaborators two simultaneous lines of research.

I) To complete the exact characterization of the dynamics for the models already analyzed:

Collaborations:	Projects:
• ENS, Lyon, France: J.-M.Maillet et al.	XXZ spin chains, sine-Gordon model and chiral Potts model with general integrable boundaries.
• LPTM, Orsay, France: V.Terras et al.	Dynamical 6-vertex and elliptic 8-vertex models.
• IMB, Dijon, France: N.Kitanine et al.	Open integrable quantum systems and out of equilibrium statistical mechanics: PASEP.

II) Generalization of SOV method for further advanced integrable quantum models:

• ENS, Lyon, France: J.-M.Maillet	Characterization of spectrum and dynamics of spin chains associated to higher rank quantum groups.

These last models should lead to the SOV tools for the solution of models like the Hubbard model of central interest both in Condensed Matter Theory and in Gauge Theory by AdS/CFT.

Main ideas to prove the determinant identities

The main identities to prove are the following:

$$\mathcal{A}_{\{z\}}[f_{\xi_+,\xi_-,\{x\}}] = \mathcal{I}_{\xi_+,\xi_-}(\{z\},\{x\}) = (-1)^L \mathcal{I}_{\tilde{\xi}_+,\tilde{\xi}_-}(\{x\},\{z\}) = (-1)^L \mathcal{A}_{\{x\}}[f_{\tilde{\xi}_+,\tilde{\xi}_-,\{z\}}].$$

$\tilde{\xi}_\pm \equiv \frac{\eta}{2} - \xi_\pm$ and $M=L$, where we have defined a generalized Izergin's determinant

$$\mathcal{I}_{\xi_+,\xi_-}(\{z\},\{x\}) = \frac{\prod_{j,k=1}^{L}(z_j^2 - x_k^2)\, \det_{1\leqslant i,j\leqslant L}\left[\sum_{\epsilon\in\{+,-\}} \epsilon \frac{(z_i+\epsilon\xi_+)(z_i+\epsilon\xi_-)}{z_i\left[(z_i+\epsilon\frac{\eta}{2})^2 - x_j^2\right]}\right]}{\prod_{j<k}(z_j^2 - z_k^2)(x_k^2 - x_j^2)}.$$

The symmetry of the generalized Izergin's determinant follows from the identity:

$$\sum_{\epsilon\in\{+,-\}} \epsilon \frac{(z+\epsilon\xi_+)(z+\epsilon\xi_-)}{z\left[(z_i+\epsilon\frac{\eta}{2})^2 - x^2\right]} = \sum_{\epsilon\in\{+,-\}} \epsilon \frac{(x+\epsilon\tilde{\xi}_+)(x+\epsilon\tilde{\xi}_-)}{x\left[(x+\epsilon\frac{\eta}{2})^2 - z^2\right]}$$

while the identity $\mathcal{A}_{\{z\}}[f_{\xi_+,\xi_-,\{x\}}] = \mathcal{I}_{\xi_+,\xi_-}(\{z\},\{x\})$ is proven multiplying and dividing $\mathcal{A}_{\{z\}}[f_{\xi_+,\xi_-,\{x\}}]$ for $\det_L \left[\mathcal{C}^X\right] = \widehat{V}(x_L,\ldots,x_1)$ and observing that:

$$\sum_{j=1}^{L} \mathcal{C}^X_{j,k} \sum_{\epsilon\in\{+,-\}} f_{\xi_+,\xi_-,\{x\}}(\epsilon z_i)\left(z_i + \epsilon\frac{\eta}{2}\right)^{2(j-1)} = \prod_{\ell=1}^{L}\left(z_i^2 - x_\ell^2\right) \sum_{\epsilon\in\{+,-\}} \epsilon \frac{(z_i+\epsilon\xi_+)(z_i+\epsilon\xi_-)}{z_i\left[(z_i+\epsilon\frac{\eta}{2})^2 - x_k^2\right]},$$

where the $L\times L$ matrix $\mathcal{C}^X$ has elements defined by $\sum_{j=1}^{L} \mathcal{C}^X_{j,k}\, \lambda^{2(j-1)} = \prod_{\substack{\ell=1\\ \ell\neq k}}^{L}(\lambda^2 - x_\ell^2)$.