FREE PARAFERMIONS

Francisco C. Alcaraz

Universidade de São Paulo, IFSC, São Carlos, SP, Brazil

Collaboration: Murray T. Batchelor (Canberra, Australia)
Zi-Zhong Liu (Chongqing, China)

1) Ising model

L Onsager, Phys. Rev. **65**, 117 (1944)

B M McCoy & T T Wu, *The Two-Dimensional Ising Model* (Dover, 2014)

- Onsager's 33 page paper is a mathematical *tour de force*!

But how did Onsager find his solution?

Much later Onsager revealed that he first diagonalised the row transfer matrix by hand: first the $2 \times \infty$ matrix, then the $3 \times \infty$ matrix, and so on.

Eventually, by the $6 \times \infty$ case, he confirmed that the $2^6 = 64$ eigenvalues were all of the form $\exp(\pm\gamma_1 \pm \cdots \pm \gamma_6)$.

That suggested an underlying product algebra which, in turn, had led to the elaborate structure of his original derivation.

This is the underlying structure of **free fermions**.

free fermions

L-site (open) Ising chain at criticality $\lambda = \lambda_c \quad = 1$

$$H = -\sum_{j=1}^{L} \sigma_j^x - \lambda \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z$$

solution in terms of free fermions (also for general λ) using the Jordan-Wigner transformation.

2^L eigenvalues of the form

$$\boxed{E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_L}$$

with (for $\lambda = \lambda_c = 1$)

$$\epsilon_k = 2\cos\frac{\pi k}{2L+1}$$

excitation spectrum

$L = 3$ quasi-particle picture

	1	2	3	3	2	1		
E_0	●	●	●	○	○	○	$-\varepsilon_1 - \varepsilon_2 - \varepsilon_3$	
E_1	●	●	○	●	○	○	$-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 = E_0 + 2\varepsilon_3$	
E_2	●	○	●	○	●	○	$-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 = E_0 + 2\varepsilon_2$	$1-particle$
E_3	○	●	●	●	○	○	$\varepsilon_1 - \varepsilon_2 - \varepsilon_3 = E_0 + 2\varepsilon_1$	
E_4	●	○	○	●	●	○	$-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = E_0 + 2\varepsilon_2 + 2\varepsilon_3$	
E_5	○	●	○	●	○	●	$\varepsilon_1 - \varepsilon_2 + \varepsilon_3 = E_0 + 2\varepsilon_1 + 2\varepsilon_3$	$2-particle$
E_6	○	○	●	○	●	●	$\varepsilon_1 + \varepsilon_2 - \varepsilon_3 = E_0 + 2\varepsilon_1 + 2\varepsilon_2$	
E_7	○	○	○	●	●	●	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = E_0 + 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3$	$3-particle$

connection to conformal field theory

For such open boundary conditions, expect (Cardy)

$$
\begin{aligned}
E_0(L) &= Le_\infty + f_\infty - \frac{\pi\zeta c}{24L} + \cdots \\
E_n(L) &= E_0(L) + \frac{\pi\zeta(x_n + r)}{L} + \cdots \qquad r = 0, 1, 2, \ldots
\end{aligned}
$$

c = central charge, x_n = scaling dimensions (critical exponents) and ζ is a scale factor. Easy to show that

$$
e_\infty = \lim_{L\to\infty} \frac{E_0(L)}{L} = -\frac{4}{\pi}, \qquad f_\infty = 1 - \frac{2}{\pi}.
$$

$$
\boxed{c = \frac{1}{2}, \quad x_\sigma = x_1 = \frac{1}{2}, \quad x_\epsilon = x_1 + x_2 = 2}
$$

T W Burkhardt & I Guim J. Phys. A 18, L33 (1985)

3) Yang-Baxter integrable spin chains

Consider the $N \times N$ matrices

$$\begin{aligned} (X)_{\ell m} &= \delta_{\ell,m+1} \pmod{N} \\ Z &= \mathrm{diag}\left(1, \omega, \omega^2, \ldots, \omega^{N-1}\right) \end{aligned}$$

with $\omega = \mathrm{e}^{2\pi i/N}$. E.g., for $N = 3$,

$$X = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}.$$

With e the identity, they satisfy

$$X^N = Z^N = e, \qquad X^\dagger = X^{N-1}, \qquad Z^\dagger = Z^{N-1},$$

$$ZX = \omega XZ.$$

There are some well studied Yang-Baxter integrable N-state quantum spin chains of the form

$$\boxed{-H = \sum_{j=1}^{L} \sum_{n=1}^{N-1} a_n \left(\lambda X_j^n + Z_j^n Z_{j+1}^{N-n} \right)}$$

$$X_j = e \otimes e \otimes \cdots \otimes e \otimes X \otimes e \otimes \cdots \otimes e$$
$$Z_j = e \otimes e \otimes \cdots \otimes e \otimes Z \otimes e \otimes \cdots \otimes e$$

where e (the unit matrix), X and Z are $N \times N$ matrices, X and Z occur in position j.

special cases

- N-state quantum Potts model

$$\boxed{a_n = 1} \tag{1}$$

- Fateev-Zamolodchikov Z_N model

$$\boxed{a_n = \frac{1}{\sin(\pi n/N)}} \tag{2}$$

- N-state superintegrable chiral Potts model

$$\boxed{a_n = \frac{2}{1 - \omega^{-n}}} \tag{3}$$

Each model reduces to the quantum Ising model for $N = 2$.

Models (1) and (2) are equivalent for $N = 3$.

Model (3) has a free fermion solution described by an Onsager algebra for general N.

4) Baxter's Z_N chain

[redacted]

A model that received very little attention up until recently was found by Baxter in 1989.

For an L-site chain this model can be written as

$$\boxed{-H = \sum_{j=1}^{L} \alpha_j X_j + \sum_{j=1}^{L-1} \gamma_j Z_j^{\dagger} Z_{j+1}}$$

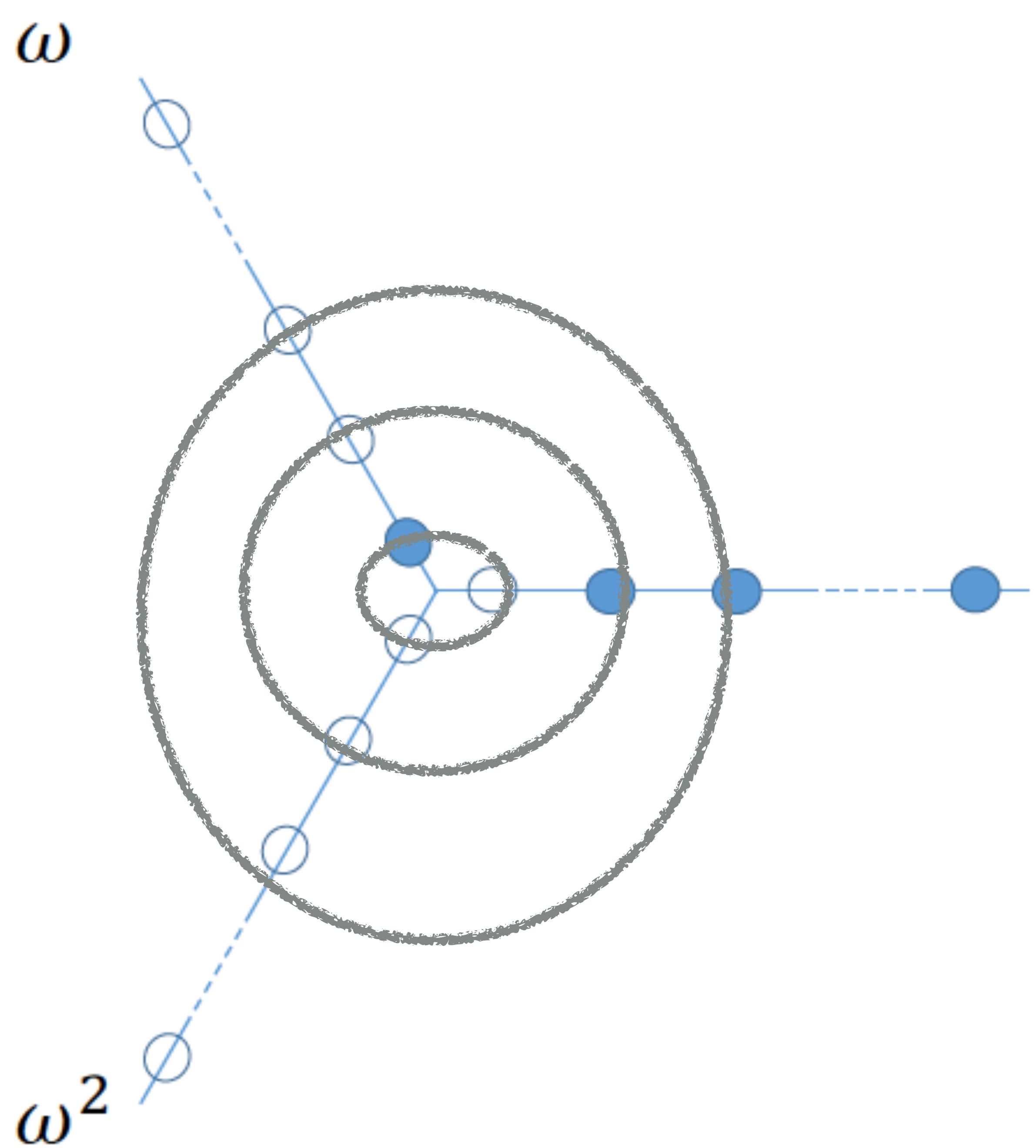
It also reduces to the quantum Ising model for $N = 2$.

This model has open boundary conditions.

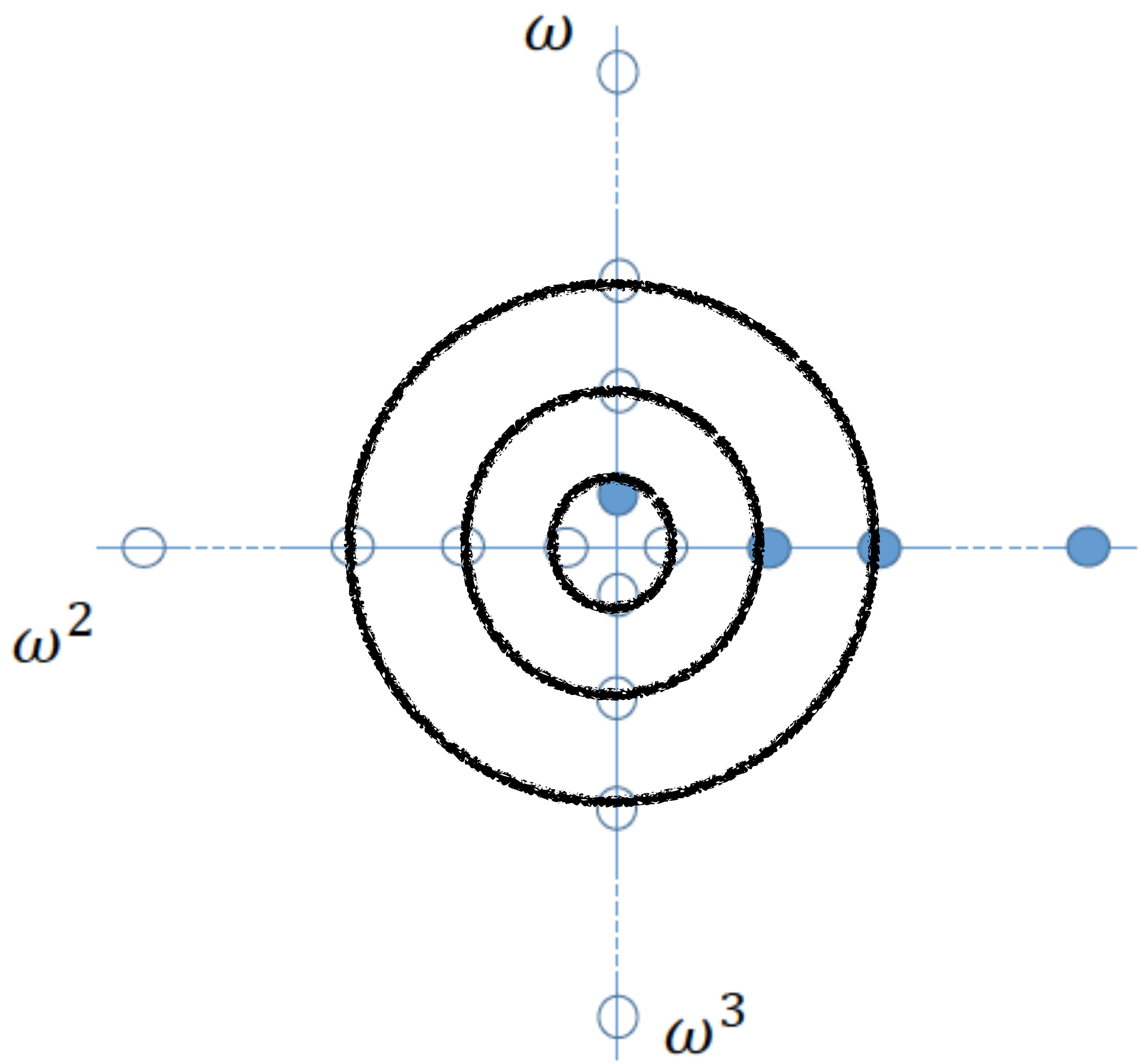
5) Free parafermions

Each index k has N possible choices, instead of "filled" or "empty" in the free fermion ($N = 2$) case. Now have a "Fermi exclusion circle".

$N = 3$ $\left(\omega = e^{2\pi i/3}\right)$



$N = 4$ $(\omega = \mathrm{i})$



NB there can be real excitations for N even, since $\omega^2 = -1$.

$$-E = \omega^{p_1}\epsilon_1 + \omega^{p_2}\epsilon_2 + \cdots + \omega^{p_L}\epsilon_L$$

- Fendley derived this result using a generalisation of the Jordan-Wigner transformation, namely the Fradkin-Kadanoff transformation to parafermionic operators originally introduced for the N-state clock models.

- Baxter (2014) and Au-Yang and Perk (2014,2016) applied Fendley's parafermionic approach to the more general τ_2 model with open boundaries.

- The chiral Potts model is related, via the τ_2 model, to the six-vertex model.

R J Baxter, J. Phys. A 47, 315001 (2014)

H Au-Yang and J H H Perk, J. Phys. A 47, 315002 (2014); arXiv:1606.06319

We consider the simple isotropic free parafermionic Hamiltonians

$$-H(\lambda) = \sum_{j=1}^{L} X_j + \lambda \sum_{j=1}^{L-1} Z_j^+ Z_{j+1}$$

Duality Transformation: $\zeta_i = X_i, \quad \eta_i = Z_i Z_{i+1}^+$

$$H = -\sum_i (\zeta_i + \lambda \eta_i)$$

$[\eta_i, \zeta_j] = 0, \text{ unless } |i-j| = 1, \text{ and } \zeta_i \eta_i = \omega \eta_i \zeta_i, \quad \eta_{i-1} \zeta_i = \omega \zeta_i \eta_{i-1}$

Duality Transformation: $\zeta_i \leftrightarrow \eta_i$

do not change the commutation relations

$$H = -\sum_i (\zeta_i + \lambda \eta_i) \longrightarrow \tilde{H} = -\sum_i (\eta_i + \lambda \zeta_i)$$

$$H(\lambda) = \lambda H(1/\lambda)$$

$\lambda = 1$ self-dual critical point for all N

The eigenspectrum

$$-E = \omega^{p_1}\epsilon_1 + \omega^{p_2}\epsilon_2 + \cdots + \omega^{p_L}\epsilon_L$$

The roots $\epsilon_1, \ldots, \epsilon_L$ obtained from the 2Lx2L Baxter/Fendley determinant

$$\begin{vmatrix} -\epsilon^{N/2} & 1 & 0 & 0 & .. & 0 & 0 \\ 1 & -\epsilon^{N/2} & \lambda^{N/2} & 0 & .. & 0 & 0 \\ 0 & \lambda^{N/2} & -\epsilon^{N/2} & 1 & .. & 0 & 0 \\ 0 & .. & .. & .. & .. & .. & 0 \\ 0 & 0 & 0 & 0 & \lambda^{N/2} & -\epsilon^{N/2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\epsilon^{N/2} \end{vmatrix} = 0.$$

$$\epsilon_j = (1 + \lambda^N + 2\lambda^{N/2}\cos(k_j))^{1/N}$$

$$\sin[(L+1)k_j] = -\lambda^{N/2}\sin(Lk_j)$$

$$\sin[(L+1)k_j] = -\lambda^{N/2}\sin(Lk_j) \qquad (j=1,\ldots,L)$$

For the critical case: $\lambda = 1$

$$k_j = \frac{2\pi}{2L+1}j \quad (j=1,\ldots,L)$$

$$\epsilon_k = \left(2\cos(\frac{\pi k}{2L+1})\right)^{2/N}$$

ground state energy at $\lambda = 1$

The ground state energy is real and given by

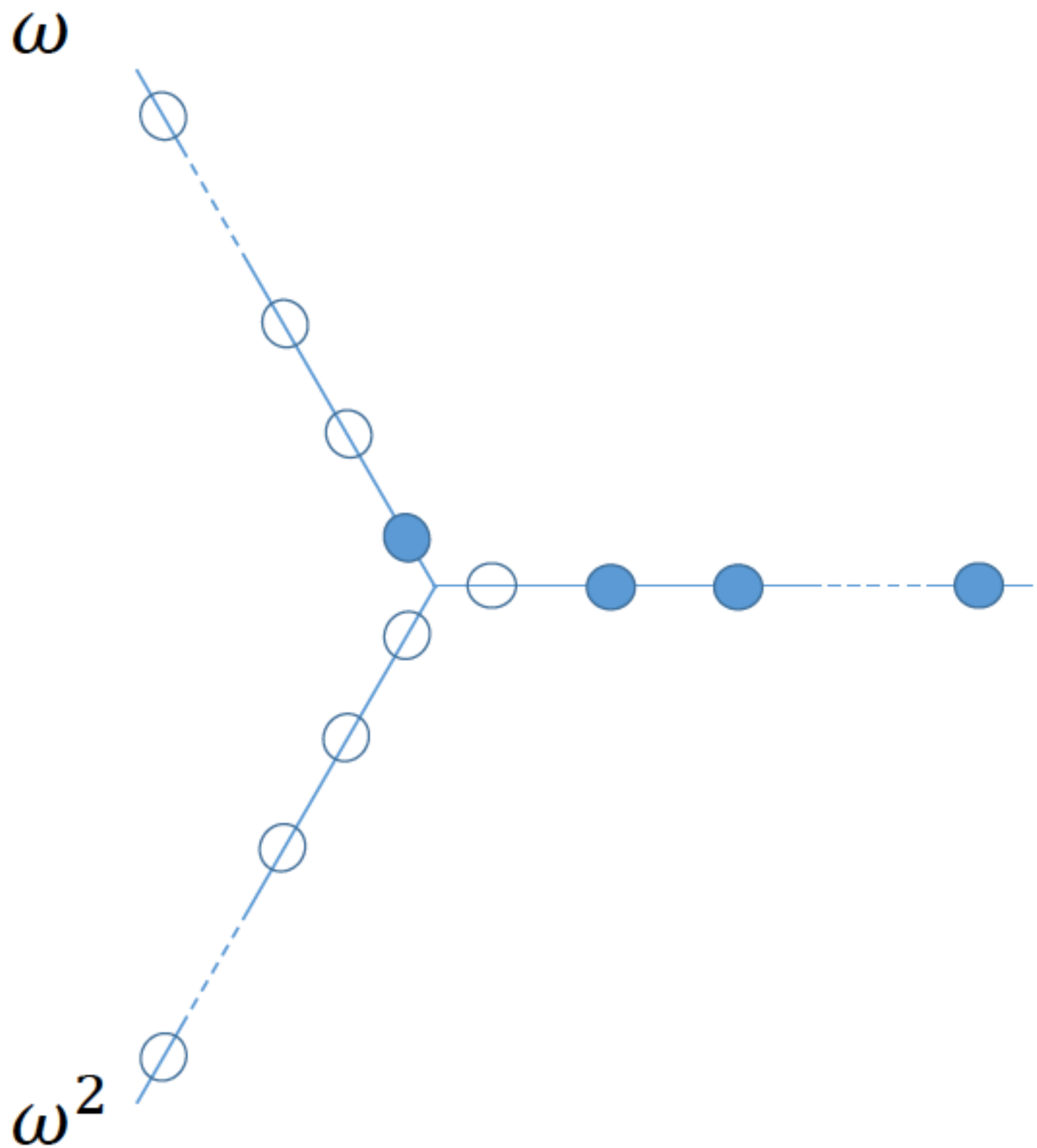
$$E_0 = -\sum_{k=1}^{L} \epsilon_k = -\sum_{k=1}^{L} \left(2\cos\frac{\pi k}{2L+1}\right)^{2/N} .$$

$$E_0(L) = Le_\infty + f_\infty + \frac{\gamma_N}{L^\nu} + O\left(\frac{1}{L^{\nu+1}}\right)$$

where $\nu = 2/N$, with

$$e_\infty = -\frac{2^\nu}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\frac{1}{N})}{\Gamma(1+\frac{1}{N})}, \qquad f_\infty = \frac{1}{2}e_\infty + 2^{\nu-1}.$$

excitations at $\lambda = 1$



Solution for general λ

$$H(\lambda) = -\sum_{j=1}^{L} X_j - \lambda \sum_{j=1}^{L-1} Z_j^+ Z_{j+1}$$

$$-E = \omega^{p_1}\epsilon_1 + \omega^{p_2}\epsilon_2 + \cdots + \omega^{p_L}\epsilon_L$$

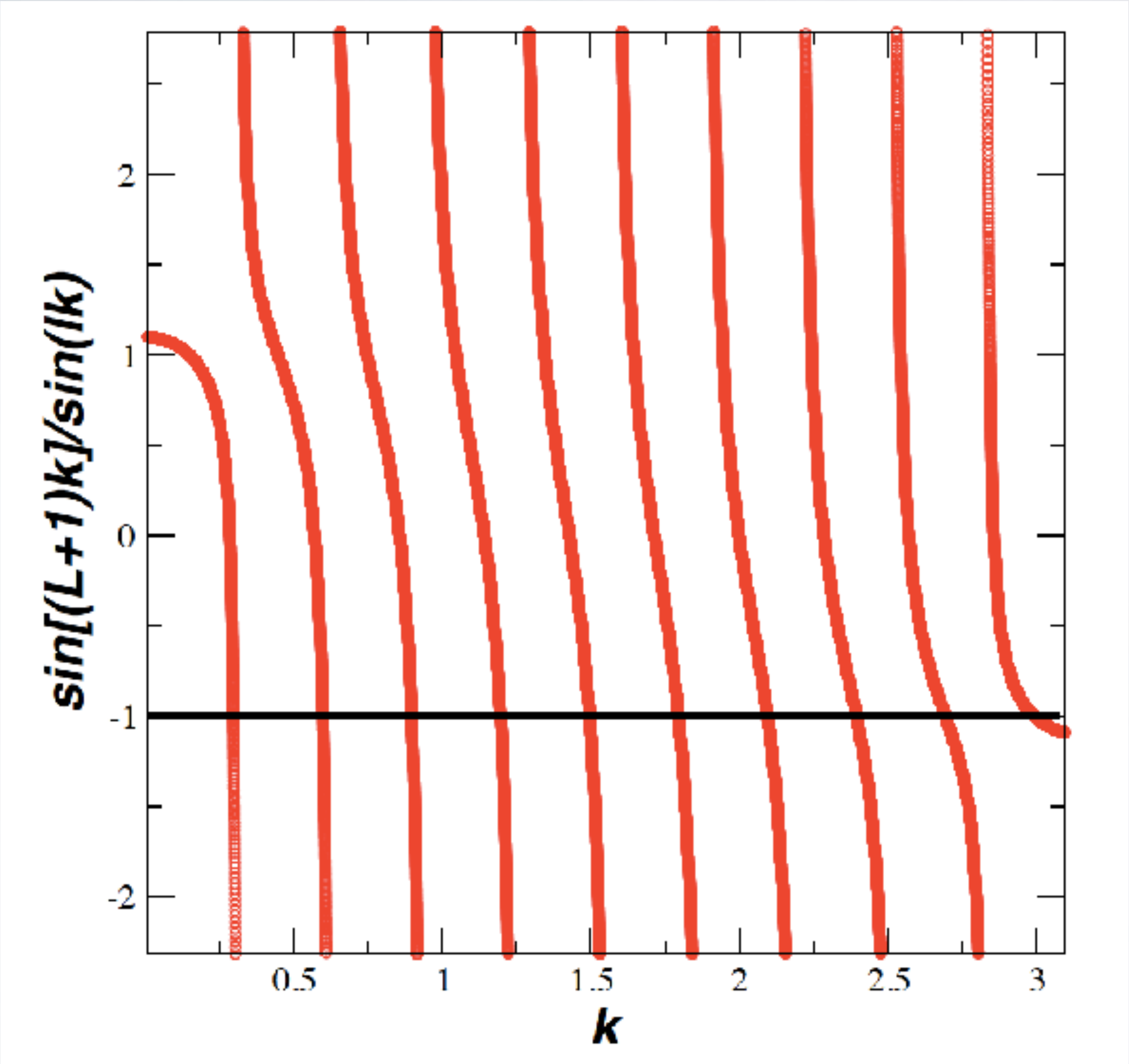
$$\epsilon_j = (1 + \lambda^N + 2\lambda^{N/2}\cos(k_j))^{1/N}$$

$$\epsilon_j = (1 + \lambda^N + 2\lambda^{N/2}\cos(k_j))^{1/N} = (1 + \lambda^{N/2})(1 + \theta^2 \sin^2(k/2)^{1/N}$$

$$\theta = \frac{4\lambda^{N/2}}{(1 + \lambda^{N/2})^2}$$

$$\sin[(L+1)k_j] = -\lambda^{N/2}\sin(Lk_j) \quad (j = 1, \ldots, L)$$

$$\frac{\sin[(L+1)k_j]}{\sin(LK_j)} = -\lambda^{N/2}(j = 1, ..., L)$$

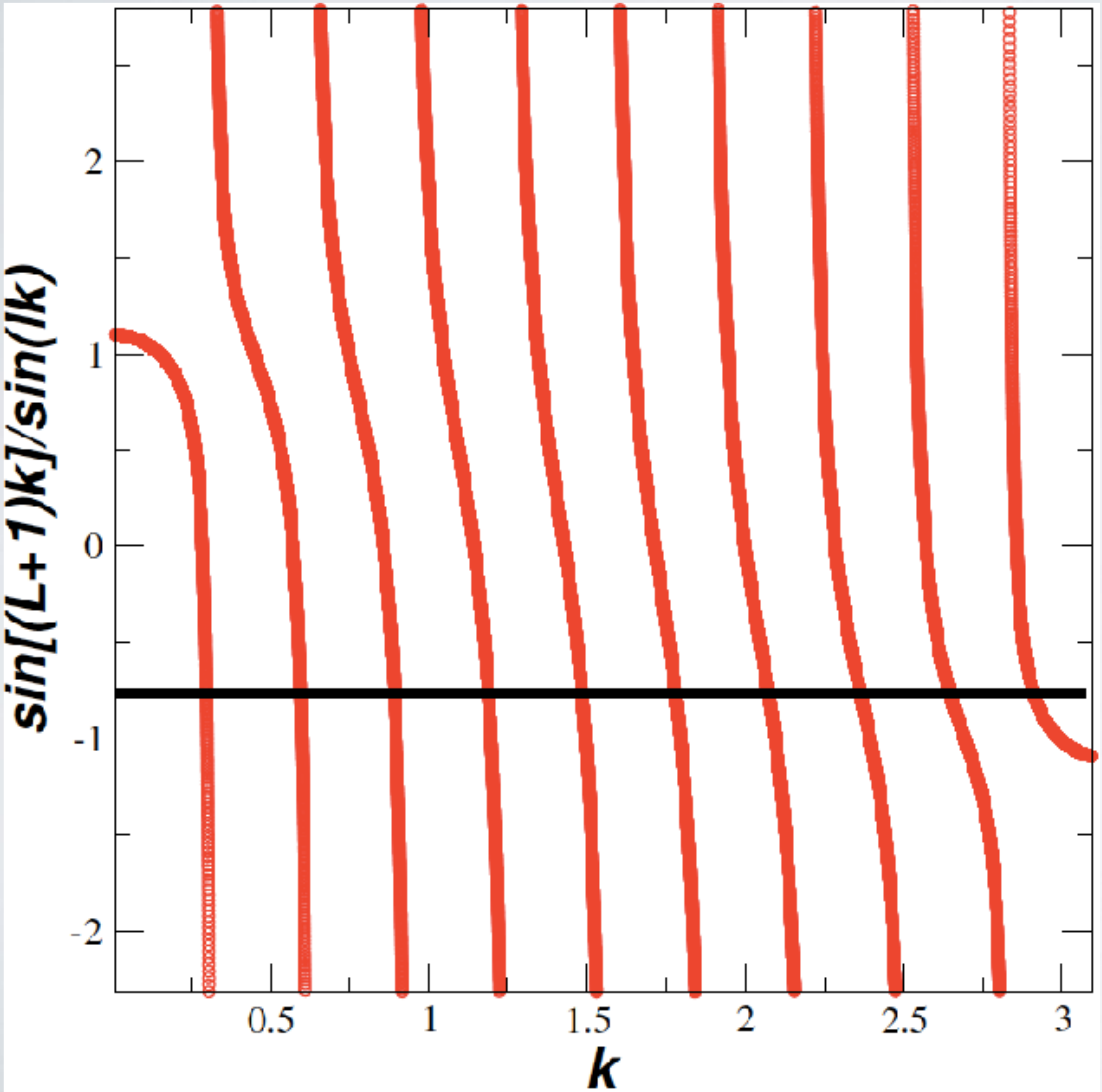


for $\lambda = 1$

$$k_j = \frac{2\pi}{2L+1} j, \quad j = 1, \ldots, L$$

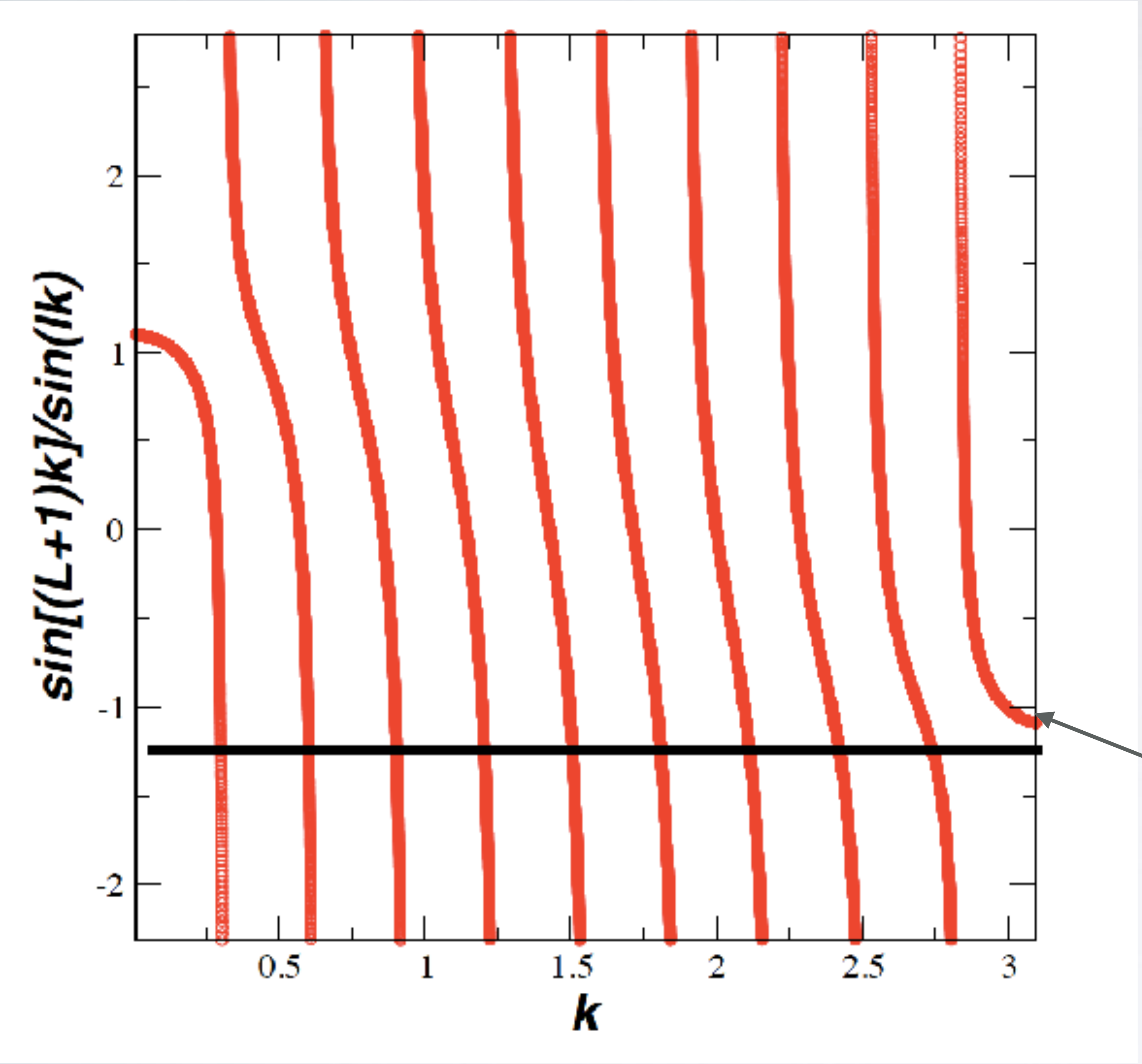
For $\lambda \neq 1$ two cases:

a) For $\lambda \leq (1+1/L)^{2/N}$



L real roots in $(0, \pi)$

b) For $\lambda > (1 + 1/L)^{2/N}$



$L - 1$ real roots in $(0, \pi)$

one real root missing

For $\lambda < 1$ write

$$Lk_j = \pi j - \pi\kappa_j + O\left(\frac{1}{L}\right), \qquad j = 1, \ldots, L$$

gives

$$\cot(\pi\kappa_j) = \frac{\lambda^{N/2} + \cos(\pi j/L)}{\sin(\pi j/L)}$$

with solution

$$\pi\kappa_j = \frac{\pi j}{2L} + \tan^{-1}\left[\frac{1-\lambda^{N/2}}{1+\lambda^{N/2}} \tan\left(\frac{\pi j}{2L}\right)\right].$$

To leading order, the roots are thus approximated for large L by

$$k_j \approx \frac{\pi j}{L} - \frac{1}{1+\lambda^{N/2}} \frac{\pi j}{L^2}, \qquad j = 1, \ldots, L.$$

For $\lambda > 1$ the complex root is of the form

$$k_L = \pi + \mathrm{i}v$$

where

$$\boxed{\sinh(L+1)v = \lambda^{N/2} \sinh Lv.}$$

Solving for large L the excitation carries energy

$$\begin{aligned} \epsilon_{k_L} &= \left(1 + \lambda^N - 2\lambda^{N/2} \cosh v\right)^{1/N} \\ &= \lambda^{1-L} \left(1 - 2\lambda^{-N} + \lambda^{-2N} + \dots\right)^{1/N}, \end{aligned}$$

- groundstate is N-fold degenerate for $\lambda > 1$.
  The gap to excitations closes as $\mathrm{e}^{-L \ln \lambda}$.

mass gaps

Can also calculate mass gaps etc. E.g., for $\lambda < 1$,

$$\boxed{E - E_0 = \varepsilon(p)\left(1 - \lambda^{N/2}\right)^{2/N}} \quad \text{as} \quad L \to \infty$$

where $p = 1, \ldots, N-1$, with

$$\varepsilon(p) = 1 - \omega^p = 1 - \cos(2\pi p/N) - \mathrm{i}\sin(2\pi p/N)\,.$$

[$N = 2$ Ising case $E - E_0 = 2(1-\lambda)$]

This result implies a correlation length exponent

$$\boxed{\nu_\perp = \frac{2}{N}}$$

$\rightarrow$ same as for the superintegrable chiral Potts model.

specific heat

$$C(\lambda, L) = -\frac{\lambda^2}{L}\frac{d^2 E_0(\lambda, L)}{d\lambda^2}$$

At the critical point $\lambda = 1$, $\quad C \sim L^{\alpha/\nu_{\|}} \quad (L \to \infty)$

We can estimate analytically that , $C \sim L^{1-2/N} \quad (L \to \infty)$

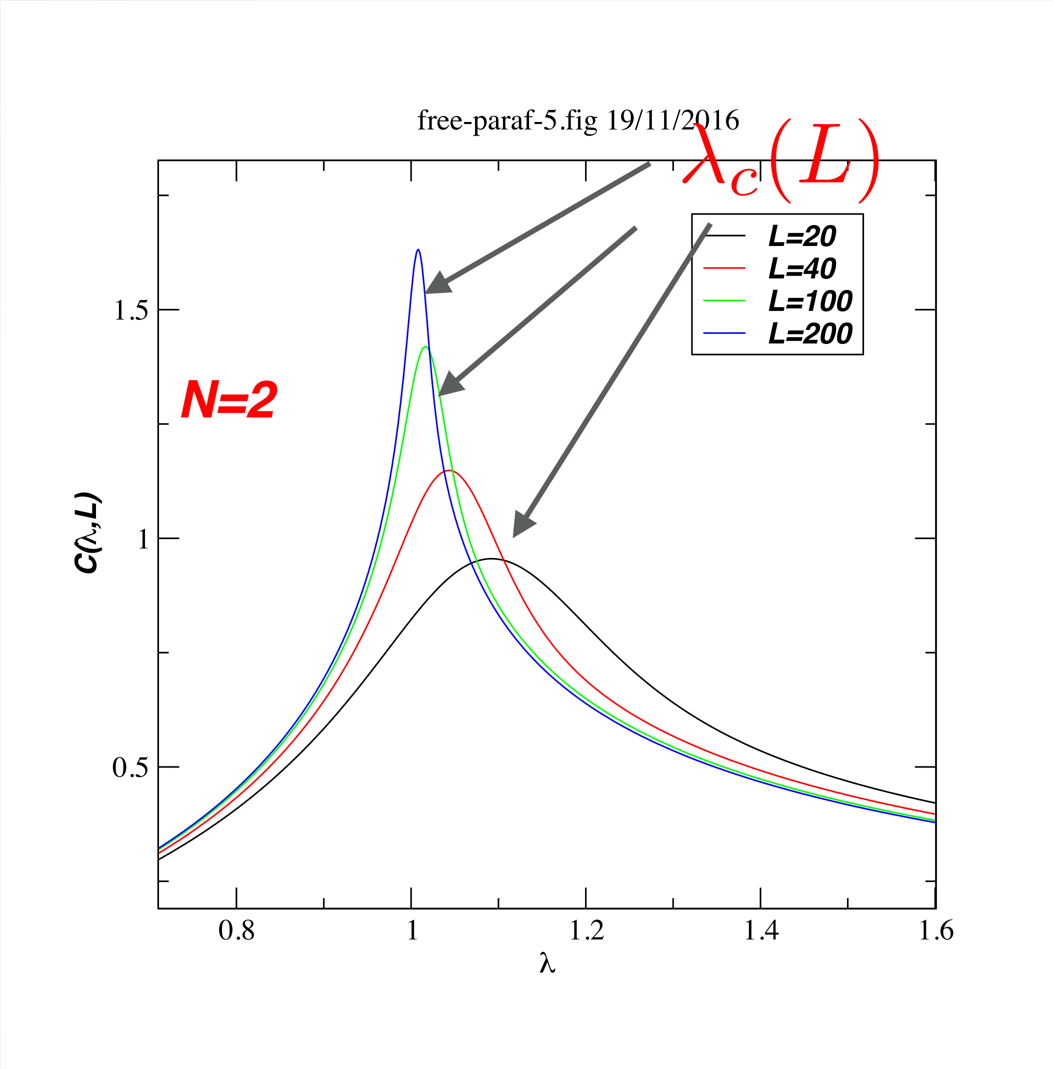
$$\frac{\alpha}{\nu_{\|}} = 1 - \frac{2}{N}$$

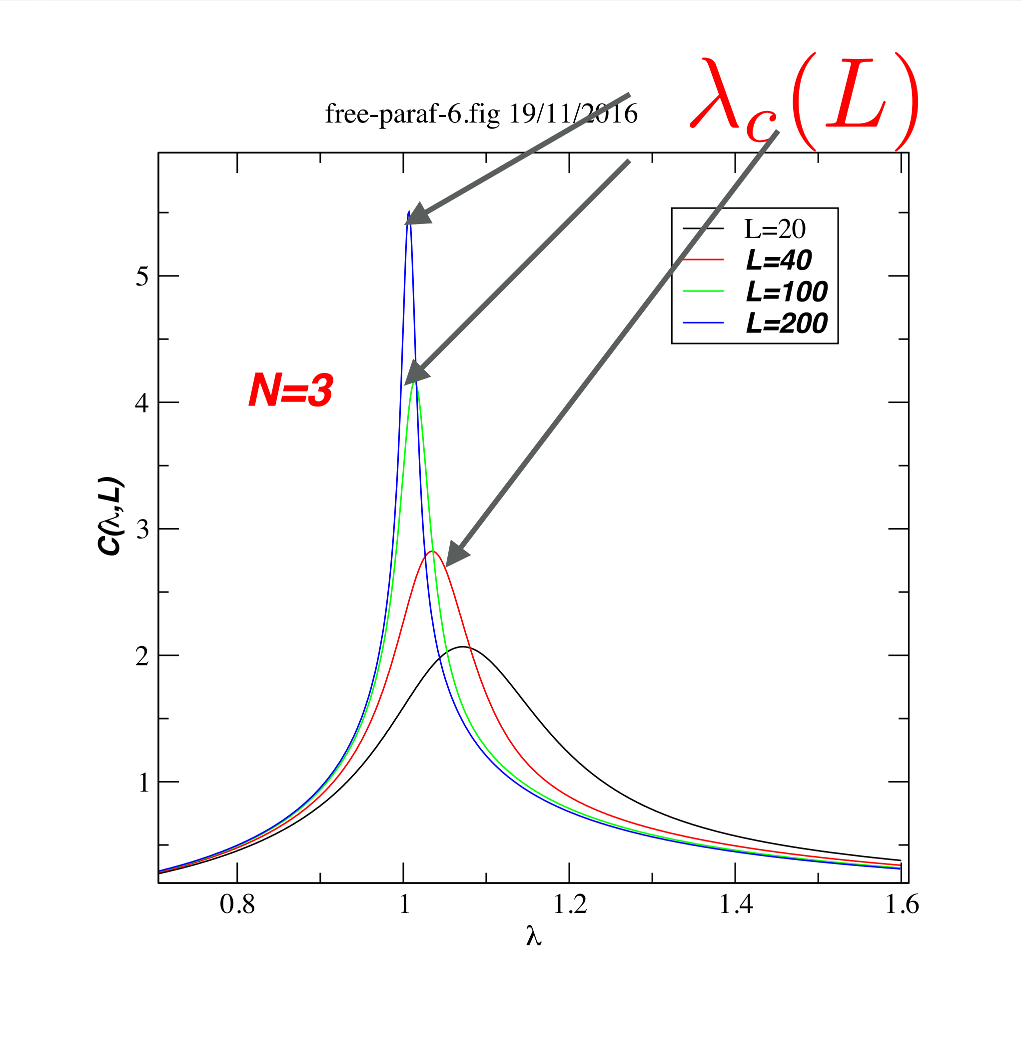
How to calculate $\nu_{\parallel}$?

$C(\lambda, L)$ shows a peak at $\lambda_c(L)$, $(\lambda_c(\infty) = 1)$

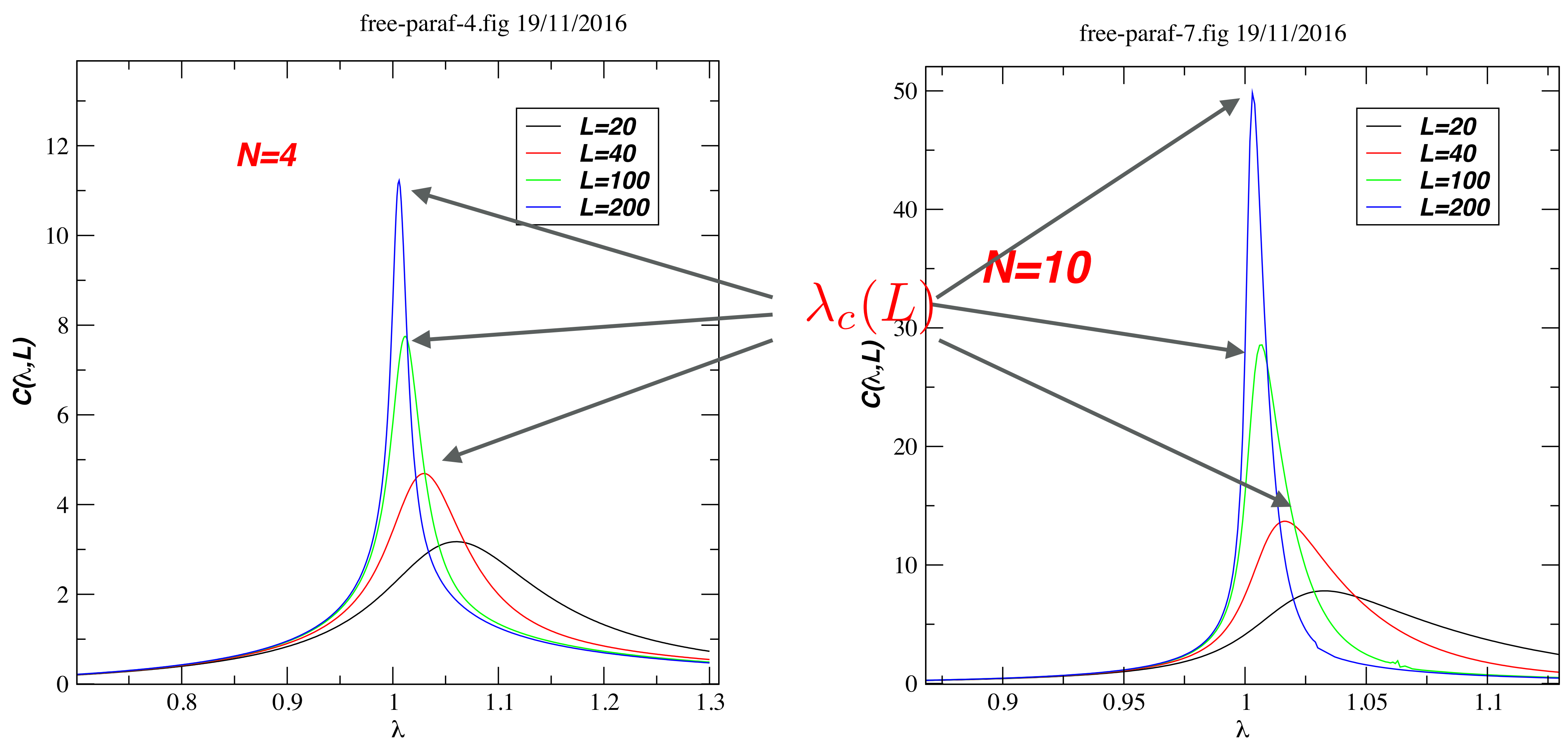
$$\lambda_c(L) - \lambda_c(\infty) \sim L^{-\nu_{\parallel}}$$

Solving numerically for all the roots, we obtain:

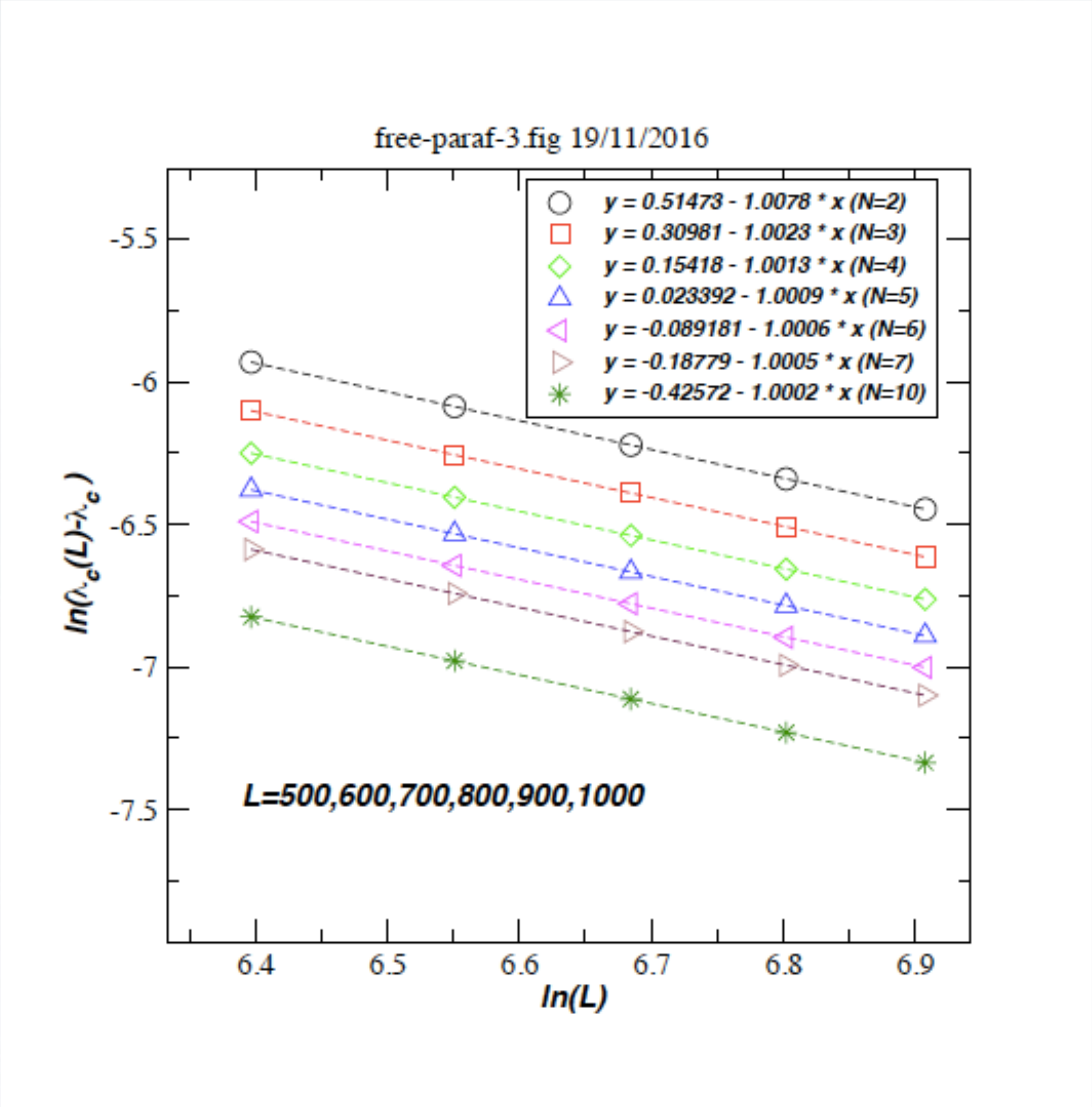




$$\lambda_c(L) - \lambda_c(\infty) \sim L^{-\nu_{\parallel}} \rightarrow \nu_{\parallel} = 1$$



$$\lambda_c(L) - \lambda_c(\infty) \sim L^{-\nu_\parallel}$$



$$\lambda_c(L) - \lambda_c(\infty) \sim L^{-\nu_\parallel} \rightarrow \nu_\parallel = 1$$

Critical exponents

$$\boxed{\alpha = 1 - 2/N}$$

$$\nu_{\|} = 1$$

$$\nu_{\perp} = 2/N = z$$

same exponent as the Z(n) Fateev-Zamolodchikov model

Fateev & Zamolodchikov (1985)

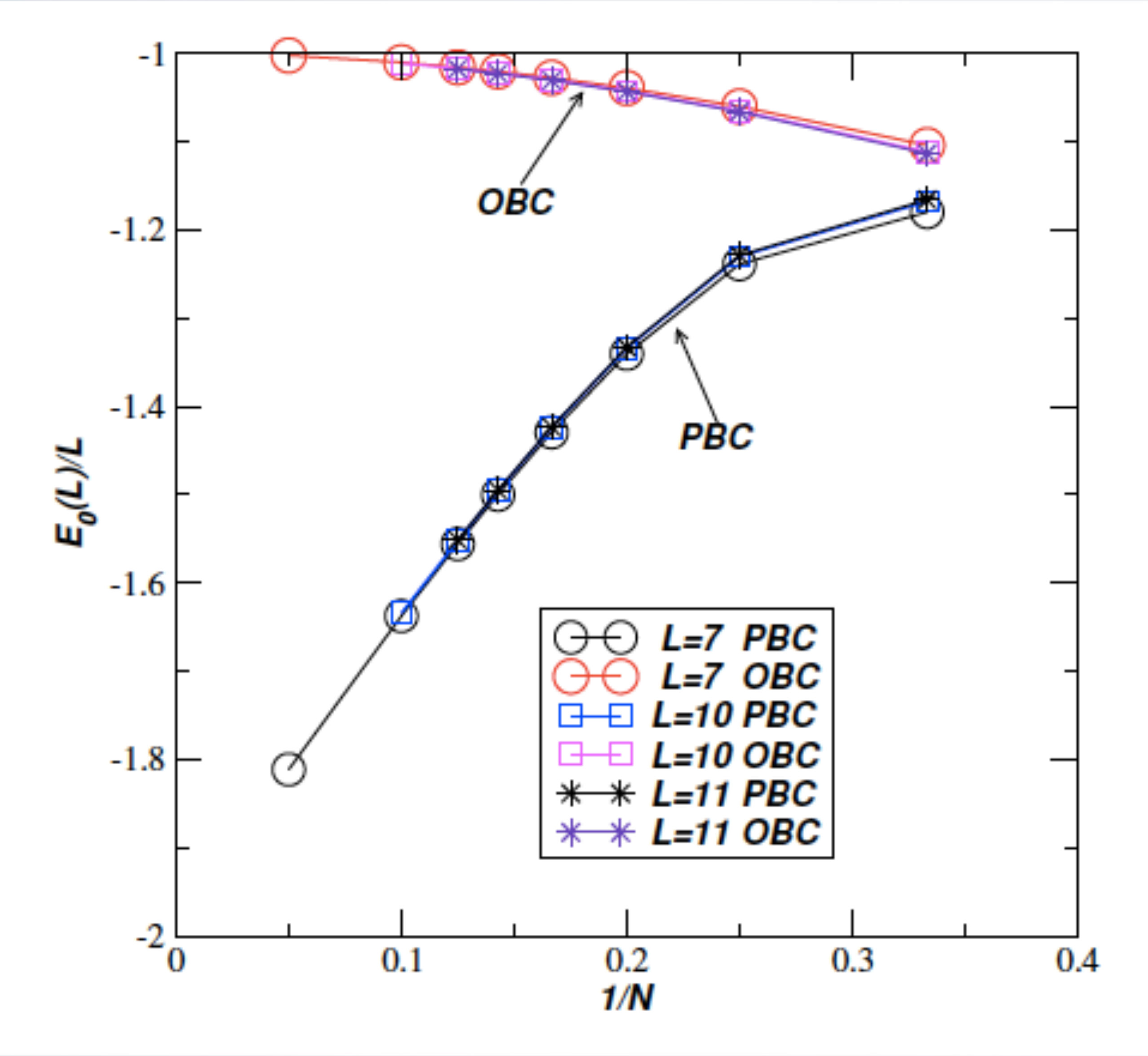
Same exponents as the Z(N) superintegrable chiral Potts Model

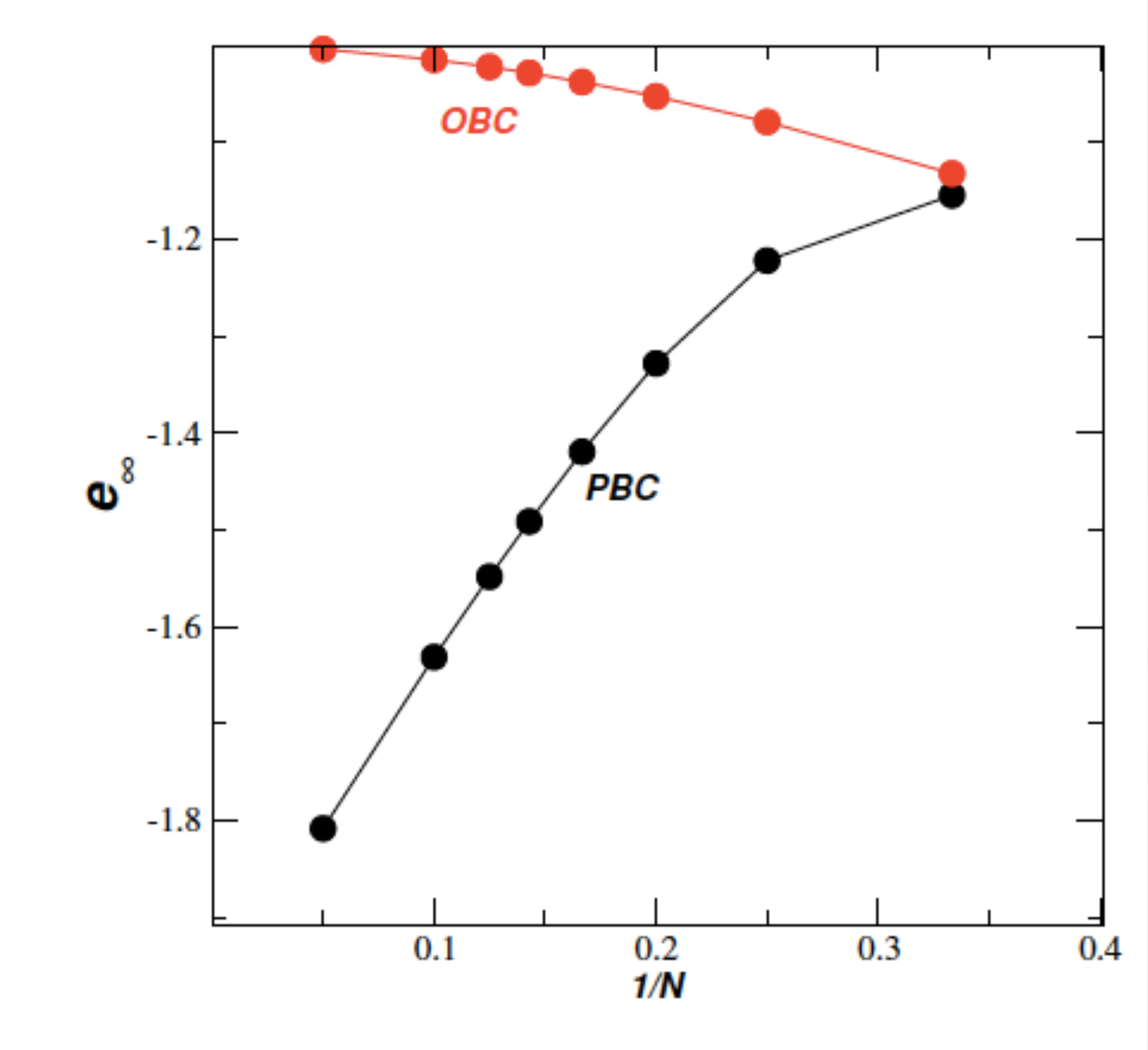
Gehlem & Rittenberg 1985, Albertini & McCoy (1989), Baxter (1989)

The periodic chain

$$H_a(\lambda) = H_{open} + aZ_L^+ Z_1$$

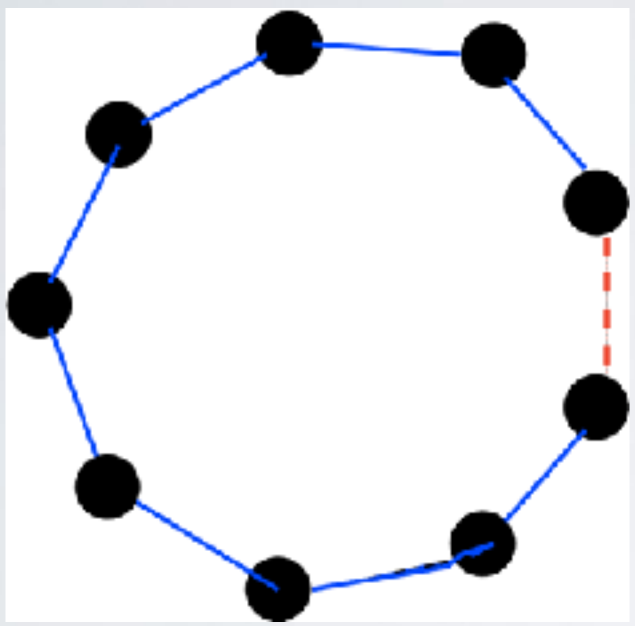
$$a = 0 \text{ (open)} \;, \quad a = 1 \text{ (periodic)}$$





SURPRISE !!!!

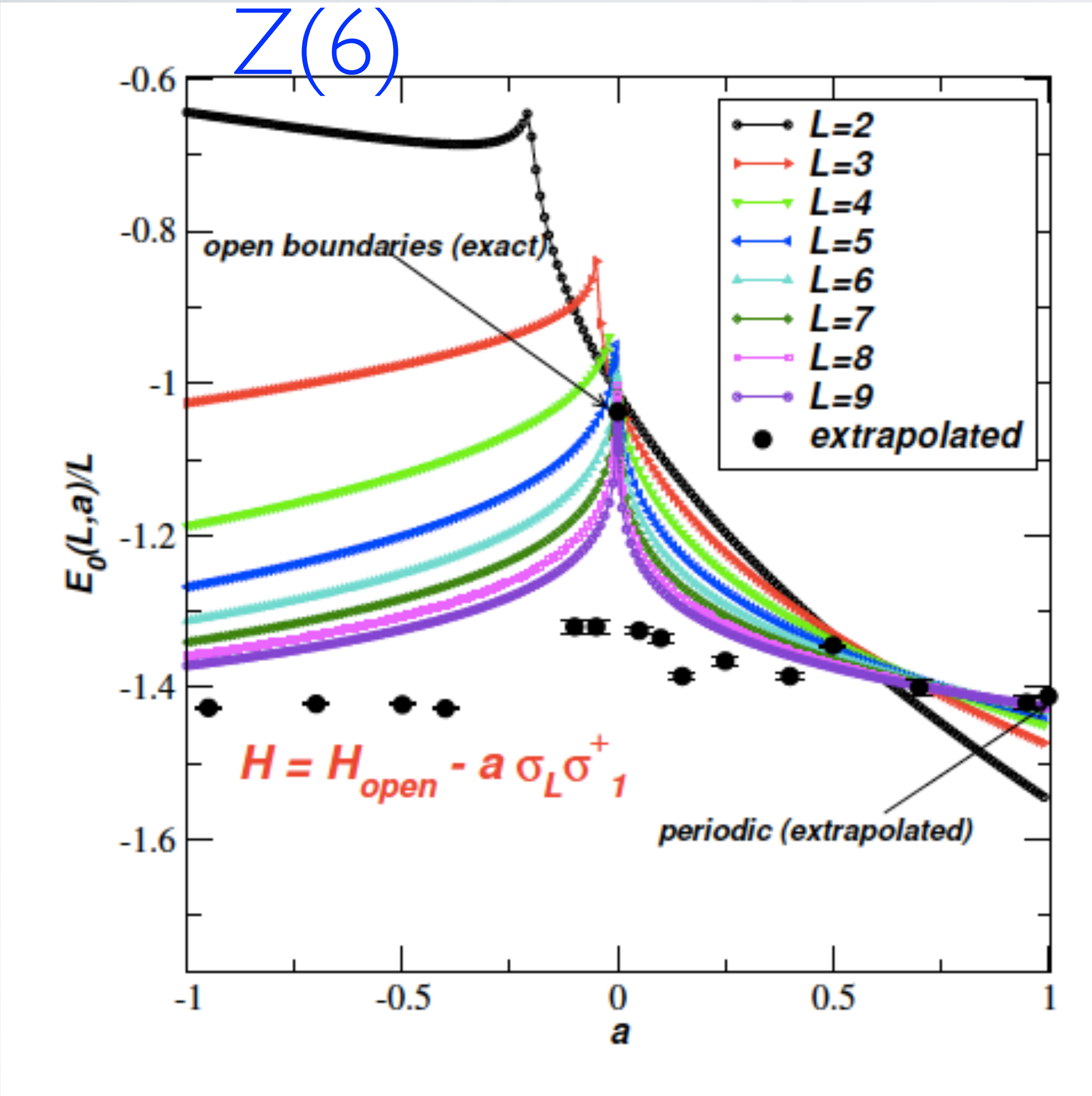
The energy decreases extensively by the simple addition of a link

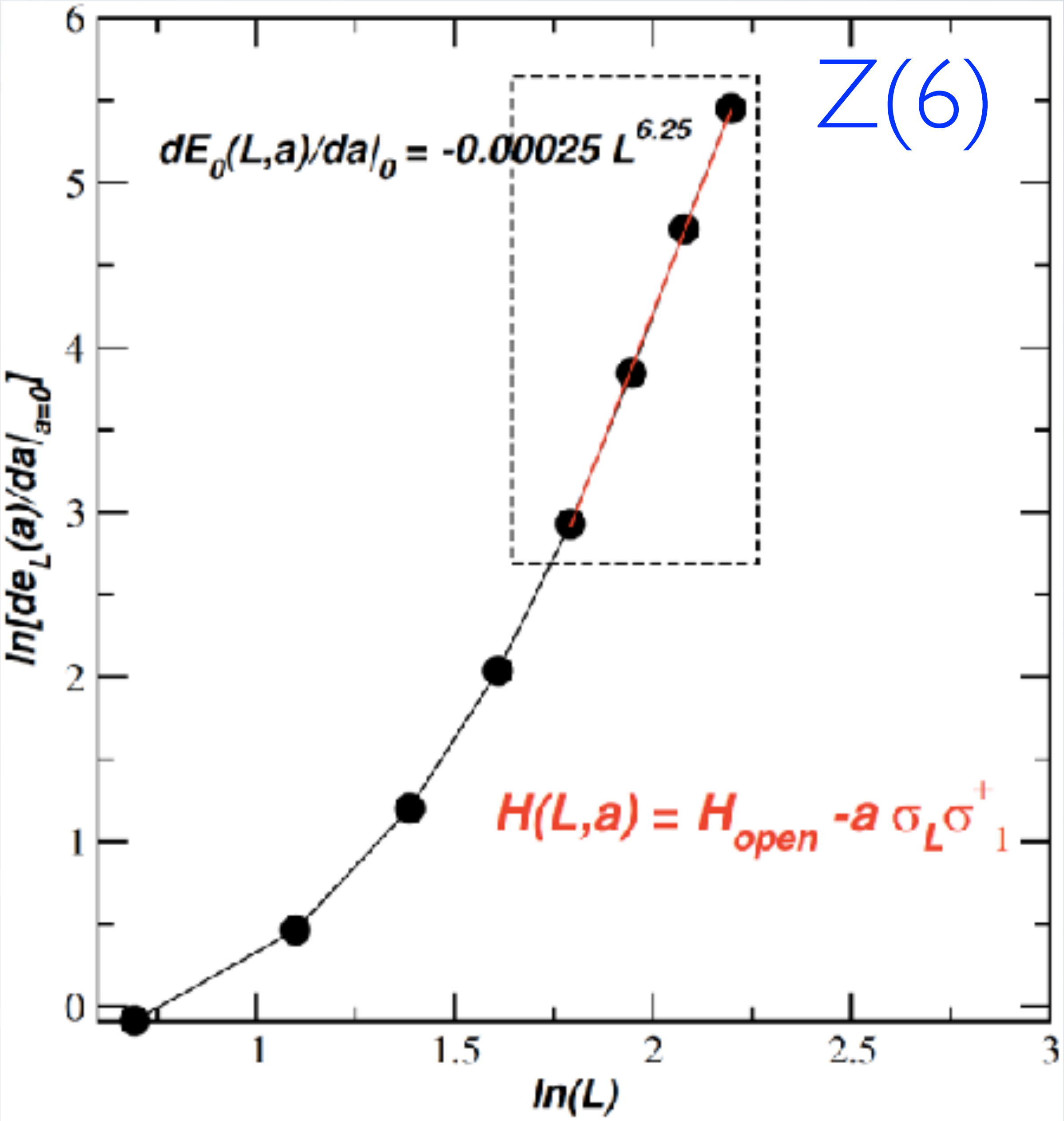


$$E_0^{\text{periodic}} = E_0^{\text{open}} - O(L) \quad !!!$$

UNEXPECTED!!!!

$$H_a(\lambda) = H_{open} + aZ_L^{+} Z_1$$





Only the free boundary is special !!!

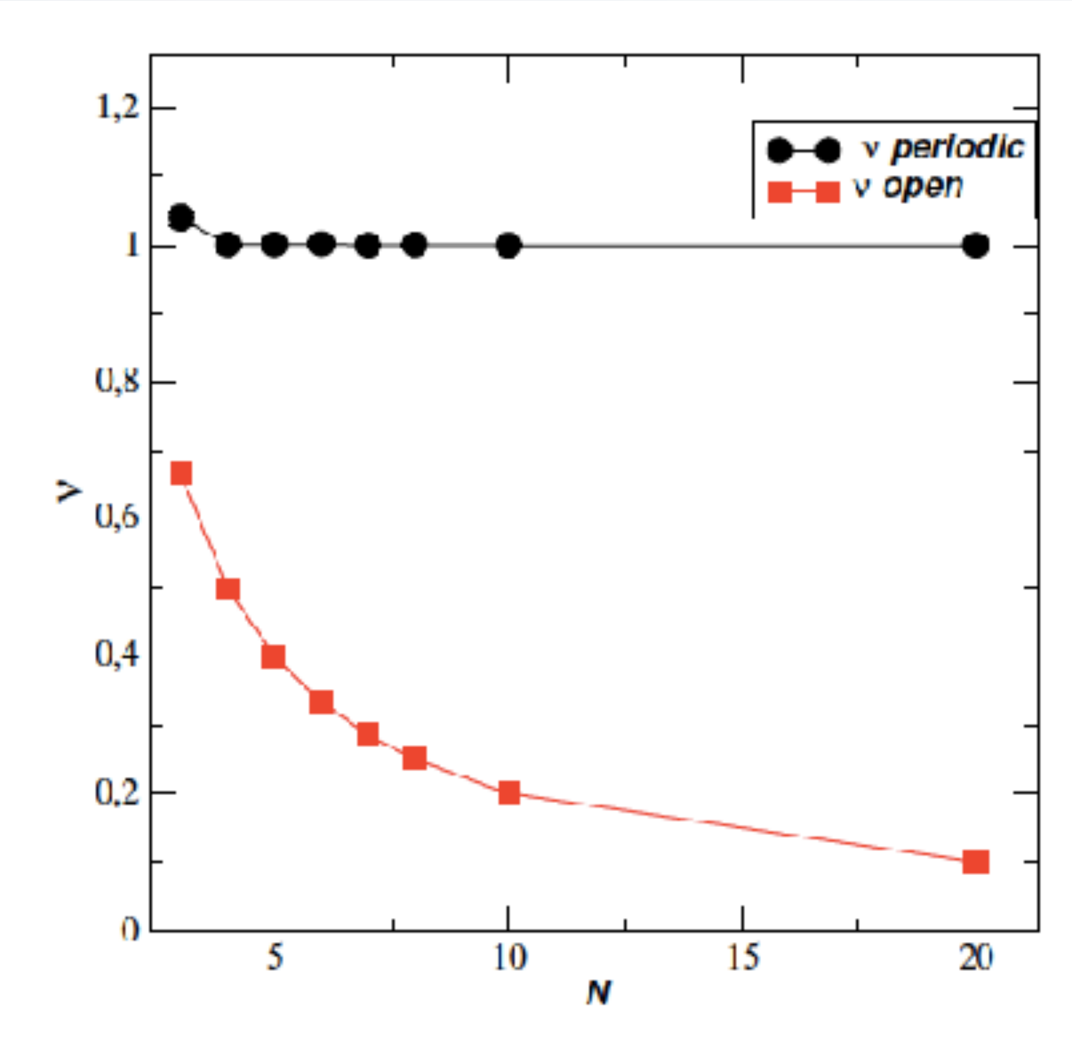
$$\frac{E_0(L)}{L} = e_\infty + \frac{f_\infty}{L} + \frac{b}{L^\gamma} + o(1/L^\gamma)$$ free boundaries $\gamma = 1 + 2/N$

$$\frac{E_0(L)}{L} = e_\infty + \frac{b}{L^\gamma} + o(1/L^\gamma)$$ periodic

N	e_∞(fit)	b(fit	γ(fit)	extr)	γ	γ_{open}
3	−1.15355	−0.6	1.68	1.70	1.68 ± 0.0	1.67
4	−1.22118	−0.7	1.89	1.92	1.90 ± 0.0	1.50
5	−1.32810	−0.6	2.02	2.02	2.02 ± 0.0	1.40
6	−1.41952	−0.5	2.05	2.01	2.03 ± 0.0	1.33
7	−1.49135	−0.4	2.06	2.02	2.04 ± 0.0	1.29
8	−1.54849	−0.4	2.06	2.03	2.04 ± 0.0	1.25
10	−1.63144	−0.3	2.06	2.02	2.04 ± 0.0	1.20
20	−1.80820	−0.1	2.07	2.03	2.05 ± 0.0	1.10

Gap

$$G_L = \mathrm{Re}\{E_1(L) - E_0(L)\} = \frac{A}{L^\nu} + o(1/L^\nu)$$



$N \to \infty \quad \nu = 1 \quad ???$

$N \to \infty \quad \nu = 0$

Specific heat

periodic $\alpha = 0$ free $\alpha = 1 - 2/N$

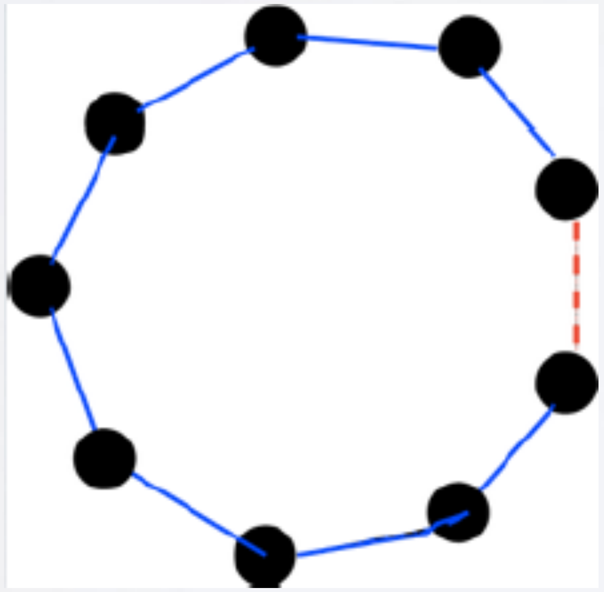
L	$N = 3$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
2	0.433013	0.248680	0.175466	0.117594	0.092118
3	0.629961	0.278889	0.189414	0.130737	0.105481
4	0.755042	0.278853	0.191451	0.135007	0.110214
5	0.840759	0.276337	0.192507	0.137145	0.112457
6	0.901140	0.274801	0.193252	0.138354	0.113684
7	0.943967	0.274056	0.193770	0.139095	0.114426
8	0.974148	0.273712	0.194129	0.139580	0.114908
9	0.995022	0.273552	0.194384	0.139914	0.115238
10	1.008975	0.273475	0.194570	0.140154	0.115475
11	1.017767	0.273437	0.194710	0.140331	0.115650
12	1.022719	0.273417	0.194816	0.140466	
13	1.024835	0.273406			
14	1.024883	0.273401			
15	1.023453				
16	1.020994				
17	1.017848				
18	1.014273				
19	1.010465				
20	1.006565				

Remarks

- Here we have an example from the class of models which are non-Hermitian, with a complex eigenspectrum.
- Nevertheless, the model has a real ground state and a remarkably simple excitation spectrum governed by the structure of **free** parafermions.
- The eigenspectrum is seen to share some critical exponents with the superintegrable chiral Potts model.

this is the first example where the "bulk" ground-state energy of a Hamiltonian depends on the boundary

In 2d vertex model there are examples: 6 vertex with domain boundary conditions V. Korepin and P. Zinn-Justin, J. Phys.A 33,7053(2000), J. Lyberg,V. Korepin, G.A.P. Ribeiro,J. Viti, J.Phys.A 59,053301(2018)



Topological effects???

OBRIGADO

THANK YOU