Integrability Aspects of Black Holes

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August 26, 2016
The Curious Case of Black Holes, Isomonodromy and CFT

- Isomonodromic integrability gives scattering coefficients of linear perturbations of black holes
- Isomonodromic tau-function is the 4-point correlator of $c=1$ CFT and might give insight into the quantum description of BHs
- Isomonodromy also gives a prescription to calculate quasinormal modes in generic BH backgrounds
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1 Black Holes, CFT and Isomonodromy
   - Black Hole Entropy via CFT
   - Scattering from Monodromies
   - Isomonodromy in $D = 4$ Kerr-de Sitter

2 Isomonodromy and CFT
   - Isomonodromic $\tau$-function and $c = 1$ CFT
   - Quantization of Isomonodromic Deformations
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This is a Black Hole!
Stationary Axissymmetric Black Hole

- Carries mass $M$ and angular momentum $J = Ma$
- Frame-dragging
- Horizon angular velocity: $\Omega_H$
Black Hole Thermodynamics and Hawking Radiation

- Black holes depend only on global charges \((M, J, Q; \Lambda)\)

\[
dM = \frac{\kappa}{8\pi G_N} dA + \Omega_H dJ + \Phi_E dQ
\]

- Classical Black hole + Quantum field = Hawking radiation

\[
T_H = \frac{\kappa}{2\pi}, \quad T_H \approx 6 \times 10^{-8} \left(\frac{M_\odot}{M}\right) K
\]

- Black holes now evaporate. What about unitarity?
- Radiation fixes black hole entropy numerical factor

\[
S_{BH} = \frac{A_H}{4G_N}
\]
Entropy as Microstate Counting in String Theory

- Microcanonical Ensemble

  \[ S = \log \Omega(M, J, Q) \]

- D1-D5-P system equivalent to 5d SUSY black hole with 3 charges

  \[ S = \log d \sim 2\pi \sqrt{Q_1 Q_5 n} - \frac{7}{4} \log(Q_1 Q_5 n) + \ldots \]

- Universality hint: Near-horizon of Extremal BHs are \( AdS_2 \times K \)

- Non-extremal BHs = breaking SUSY
Near-Horizon Extremal Kerr (NHEK) Metric

- Extremal Kerr Properties

\[ r_{\pm} = a = M, \quad S = 2\pi M^2 = 2\pi J, \quad T_H = 0, \quad \Omega_H = \frac{1}{2M} \]

- Near-horizon limit

\[ r = \frac{\hat{r} - M}{\lambda M}, \quad t = \frac{\lambda \hat{t}}{2M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M}, \]

- When \( \lambda \to 0 \), we get \( \text{AdS}_2 \ltimes S^1 \) for fixed \( \theta \)

\[ ds^2 = 2\Omega^2(\theta) J \left[ -r^2 dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \Lambda^2(\theta)(d\phi + r dt)^2 \right] \]
Kerr/CFT Correspondence

- ASG = Allowed diffeos / Trivial diffeos (Brown, Henneaux 1986)
- Brown-Hennaux perturbations $\delta g_{ab} = h_{ab}$ preserving metric boundary conditions are generated by a Virasoro algebra
  
  $$[L_n, L_m] = (n - m)L_{n+m} + Jm(m^2 - 1)\delta_{n+m}$$

- Corresponds to a chiral thermal CFT with temperature $T_L = \frac{1}{2\pi}$ and central charge $c = 12J$
- Cardy formula for CFT entropy reproduces black hole entropy (Guica et al 2009)
  
  $$S_{CFT} = \frac{\pi^2}{3}c_LT_L = 2\pi J = S_{BH}$$
Kerr/CFT Away From Extremality

- Wave equation for $M\omega \ll 1$ and $r\omega \ll 1$ also presents conformal symmetry (Castro et al 2010)
- Hypergeometric scattering amplitudes match SL(2, $\mathbb{C}$) symmetry of dual CFT
- For the Kerr black hole, we can write (Castro et al 2013)

$$|\mathcal{T}|^2 = \frac{\sinh 2\pi (\omega_L + \omega_R) \sinh(2\pi \alpha_{\text{irr}})}{\sinh \pi (\omega_L - \alpha_{\text{irr}}) \sinh \pi (\omega_R + \alpha_{\text{irr}})}$$

- For low-frequencies and $\ell \neq 0$

$$i\alpha_{\text{irr}} = \ell - 2M^2 \omega^2 f(\ell) + O(\omega^3)$$

suggesting no simple CFT description
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Linear Perturbation of Gravitational Systems

- Gravity \((M, g)\) + Matter field \(\Phi\)

\[
S = \frac{1}{16\pi G} \int_M d^D x \sqrt{-g} \left( R - 2\Lambda \right) + \int_M d^D x \sqrt{-g} \mathcal{L}_m(\Phi, \nabla\Phi)
\]

- Linear perturbation of equations of motion

\[
g_{ab} = g_{ab}^{BG} + h_{ab}, \quad \Phi = \Phi^{BG} + \phi
\]

- \(D = 4\) Petrov Type D solutions:
  Teukolsky master equation for spin \(s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2\)
Scalar Field Perturbation

- Non-minimally coupled massless scalar field $\phi(x)$

\[
(\nabla^2 + \xi R)\phi(x) = 0, \quad \nabla^2 \phi \equiv \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b \phi)
\]

- Separable solutions: $\phi(t, r, \theta, \varphi) = e^{-i\omega t} e^{im\varphi} S_{\omega \ell m}(\theta) R_{\omega \ell m}(r)$

- Radial and Angular equations

\[
\partial_r (P_r(r) \partial_r R_{\omega \ell m}) - Q_r(r) R_{\omega \ell m} = 0
\]

\[
\partial_\theta (P_\theta(\theta) \partial_\theta S_{\omega \ell m}) - Q_\theta(\theta) S_{\omega \ell m} = 0
\]

- Angular eigenvalues from angular equation
Complex ODEs and Monodromy

- **Self-adjoint** radial equation

\[
\partial_z (U(z) \partial_z \phi(z)) - V(z) \phi(z) = 0, \quad z \in \mathbb{CP}^1
\]

- Regular singular points \( \{ z_i \}, \ i = 1, \ldots, n \).

- Ingoing and Outgoing solutions

\[
\phi_i^\pm (z) = (z - z_i)^{\pm \theta_i / 2} \left( 1 + O(z - z_i) \right)
\]

- Singular points = Branch points \( \Rightarrow \) Monodromy

\[
\phi_i^\pm (ze^{2\pi i}) = e^{\pm i\pi \theta_i} \phi_i^\pm (z)
\]
Monodromies and Gauge Connection

- Gauge connection formulation

\[ (\partial_z - A(z)) \Phi(z) = 0 , \]

\[ A(z) = \begin{pmatrix} 0 & U^{-1} \\ V & 0 \end{pmatrix} , \quad \Phi(z) = \begin{pmatrix} \phi_1 & \phi_2 \\ U \partial_z \phi_1 & U \partial_z \phi_2 \end{pmatrix} \]

- Monodromy matrix

\[ \Phi_\gamma(z) = \mathcal{P} \exp \left( \oint_\gamma A \right) \Phi(z) =: \Phi(z) M_\gamma \]
Monodromies and Frobenius solutions

- Loop around only one pole $z = z_i \Rightarrow \Phi_{\gamma_i} = \Phi M_i$
- Loop enclosing all poles gives monodromy identity

$$M_1 M_2 \ldots M_n = 1$$

- General Frobenius solution

$$\Phi(z) = \Phi_i(z) g_i$$

$$= \left( \Phi_0^i + O(z - z_i) \right) \begin{pmatrix} (z - z_i)\theta_i/2 & 0 \\ 0 & (z - z_i)^{-\theta_i/2} \end{pmatrix} g_i$$

- Monodromy matrix in arbitrary basis

$$M_i = g_i^{-1} \begin{pmatrix} e^{i\pi \theta_i} & 0 \\ 0 & e^{-i\pi \theta_i} \end{pmatrix} g_i$$
Scattering Amplitudes and Connection Matrix

- Change of basis matrix = Connection matrix

\[ M_{ij} = \Phi_i^{-1} \Phi_j = g_i g_j^{-1} \]

\[ M_{ij} = \begin{pmatrix} \frac{1}{T} & \frac{R^*}{T^*} \\ \frac{R}{T} & \frac{1}{T^*} \end{pmatrix}, \quad |R|^2 + |T|^2 = 1 \]

If we define

\[ m_{ij} = \text{Tr} \ M_i M_j = 2 \cos \pi \sigma_{ij} \]

then

\[ |T|^2 = \left| \frac{\sin \pi \theta_i \sin \pi \theta_j}{\cos \pi (\theta_i - \theta_j) - \cos \pi \sigma_{ij}} \right| \]

(Castro et al 1304.3781, C. da Cunha and FN 1404.5188)
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$D = 4$ Kerr-de Sitter Black Hole

\[
d s^2 = -\frac{Q(r)}{r^2 + p^2} (dt + p^2 d\phi)^2 + \frac{P(p)}{r^2 + p^2} (dt - r^2 d\phi)^2 \\
+ \frac{r^2 + p^2}{Q(r)} dr^2 + \frac{r^2 + p^2}{P(p)} dp^2
\]

\[
P(p) = -\frac{\Lambda}{3} p^4 - \left(1 - \frac{\Lambda a^2}{3}\right) p^2 + a^2
\]

\[
Q(r) = -\frac{\Lambda}{3} r^4 + \left(1 - \frac{\Lambda a^2}{3}\right) r^2 - 2Mr + a^2
\]
Kerr-dS Radial Equation

- Radial equation

\[ \partial_r(Q(r)\partial_r R(r)) + \left(-4\Lambda \xi r^2 + \frac{(\omega(r^2 + a^2) - am)^2}{Q(r)}\right) R(r) = C_\ell R \]

- Frobenius coefficients

\[ \rho_i^\pm = \pm i \left(\frac{\omega(r_i^2 + a^2) - am}{Q'(r_i)}\right), \quad i = 1, \ldots, 4 \]

\[ \rho_\infty^\pm = \frac{3}{2} \pm \frac{1}{2} \sqrt{9 - 48\xi} \]

- \( \theta_\infty = \sqrt{9 - 48\xi} \) is an integer when \( \xi = 0, \frac{5}{48}, \frac{1}{6}, \frac{3}{16} \)

- For conformal coupling \( \xi = \frac{1}{6} \), infinity becomes a regular point
Heun Equation for Conformally Coupled Case

- Heun equation (4 regular singular points)

\[ y'' + \left( \frac{1 - \theta_0}{z} + \frac{1 - \theta_1}{z - 1} + \frac{1 - \theta_{t_0}}{z - t_0} \right) y' + \]

\[ + \left( \frac{1 + \theta_\infty}{z(z - 1)} - \frac{t_0(t_0 - 1)K_0}{z(z - 1)(z - t_0)} \right) y = 0 \]

- Frobenius coefficients

\[ \theta_k = \frac{i}{2\pi} \left( \frac{\omega - \Omega_k m}{T_k} \right), \quad k = 0, 1, t_0, \infty \]
Four-point Monodromy Group

\[ \det M_i = 1, \quad m_i \equiv \text{Tr } M_i = 2 \cos \pi \theta_i, \]

\[ M_\infty M_1 M_t M_0 = 1 \]
Representation of 4-point Monodromy Group

- 3 composite traces

\[ m_{ij} = \text{Tr}(M_iM_j) = 2\cos(\pi\sigma_{ij}), \quad i, j = 0, 1, t \]

- Only 2 are independent because of Fricke-Jimbo relation

\[
W_4(m_1, m_2, m_3, m_{13}, m_{23}, m_{12}, m_4) \equiv \\
\quad m_{13}m_{23}m_{12} + m_{13}^2 + m_{23}^2 + m_{12}^2 \\
- m_{13}(m_2m_4 + m_1m_3) - m_{23}(m_1m_4 + m_2m_3) - m_{12}(m_3m_4 + m_1m_2) \\
+ m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_1m_2m_3m_4 - 4 = 0
\]

- Monodromy representations are parametrized by two composite traces

\((\sigma_{0t}, \sigma_{1t}), (\sigma_{0t}, \sigma_{01})\) or \((\sigma_{1t}, \sigma_{01})\)
Isomonodromic System and Apparent Singularity

- Fuchsian System with 4 singular points

\[ \partial_z \mathcal{Y}(z) = A(z) \mathcal{Y}(z), \quad A(z) = \sum_{i=1}^{3} \frac{A_i}{z - z_i}, \]

with \( \mathcal{Y}(z) = (y_1(z) \ y_2(z))^T \)

- Component \( y_1 \) obeys the ODE

\[ \begin{align*}
\partial_z^2 y - (\partial_z \log A_{12} + \text{Tr} A(z)) \partial_z y \\
+ (\det A(z) - \partial_z A_{11} + A_{11} \partial_z \log A_{12}) y = 0
\end{align*} \]
Isomonodromic System and Apparent Singularity

- Apparent singularity at \( z = \lambda \) if
  \[
  A_{12}(z) = k \frac{z - \lambda}{z(z - 1)(z - t)}, \quad k \in \mathbb{C}
  \]

- *Deformed* Heun equation with one apparent singularity
  \[
  \partial_z^2 y + \left( \frac{1 - \theta_0}{z} + \frac{1 - \theta_1}{z - 1} + \frac{1 - \theta_t}{z - t} - \frac{1}{z - \lambda} \right) \partial_z y
  + \left( \frac{\kappa}{z(z - 1)} - \frac{t(t - 1)K}{z(z - 1)(z - t)} + \frac{\lambda(\lambda - 1)\mu}{z(z - 1)(z - \lambda)} \right) y = 0
  \]

- \( \lambda(t_0) = t_0 \) and \( \mu_0 = -K_0/\theta_t \) for our Heun
Isomonodromic Hamiltonian System

- $z = \lambda$ is an apparent singularity if

\[
K(\lambda, \mu, t) = \frac{1}{t(t-1)} \left[ \lambda(\lambda - 1)(\lambda - t)\mu^2 - \{\theta_0(\lambda - 1)(\lambda - t) + \theta_1 \lambda(\lambda - t) + (\theta_t - 1)\lambda(\lambda - 1)\} \mu + \kappa(\lambda - t) \right]
\]

- Garnier System

\[
\frac{d\lambda}{dt} = \frac{\partial K}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial K}{\partial \lambda}
\]

generates isomonodromic flow $(\lambda(t), \mu(t), K(\lambda, \mu, t))$

- Second-order equation for $\lambda(t) = \text{Painlevé VI}$
Painlevé VI $\tau$-function

- **Definition of $\tau$-function**
  $$K(\lambda, \mu, t) = \frac{d}{dt} \log(f(t)\tau(t, \{\theta_i, \sigma_{ij}\}))$$

- **$\tau$-function asymptotics (Jimbo ’82)**
  $$\tau(t) \propto t^{\sigma^2/4-(\theta_0-\theta_t)^2/4}[1 + sC_-t^{1-\sigma} + O(t^{1+\sigma}, t)], \quad 0 < \text{Re} \sigma < 1$$

- $(\sigma, s)$ are two integration constants related to monodromies $(\sigma_{0t}, \sigma_{1t})$
The auxiliary function

\[ \zeta(t) \equiv t(t - 1) \frac{d}{dt} \log \tau(t, \{\theta_i\}) \]

obeys an equivalent form of Painlevé VI

Initial conditions can be inverted to obtain \( \sigma \)

\[ \zeta(t) \bigg|_{t=t_0} = t_0 \theta_t \theta_1 + (t_0 - 1) \theta_0 \theta_t + t_0(t_0 - 1) K_0, \]

\[ \left. \frac{d\zeta(t)}{dt} \right|_{t=t_0} = -(\theta_t + \kappa_1) \theta_t = -\theta_t(\theta_t + 1), \]

and determine \((\sigma, s)\) (Carneiro da Cunha and FN '15).
Near-Extremal Kerr-dS Quasinormal modes

- **Quasinormal mode** = Pole of Transmission Amplitude

\[
\sigma_{0t} = \theta_0 - \theta_t + 2N, \quad N \in \mathbb{Z}
\]

- From isomonodromy, we get the result

\[
\frac{\omega_{N}^{\pm} - \Omega_H m}{\kappa_H} = -i(N + \frac{1}{2}) \pm \sqrt{\tilde{K}_0 - \frac{1}{4}}
\]

with

\[
\tilde{K}_0 = \frac{L^2}{(3r_H + r_-)(r_H - r_-)} \left[ -\frac{2r_H^2}{L^2} + C_\ell (m\Omega_H) - \frac{2i\chi^2}{3r_H + r_-} m\Omega_H r_H (r_H + r_-) \right]
\]
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Consider Fuchsian system with $n$ singularities

$$
\partial_z \Phi = A(z) \Phi, \quad A(z) = \sum_{\nu=1}^{n} \frac{A_\nu}{z - a_\nu}, \quad \sum_{\nu=1}^{n} A_\nu = 0,
$$

$$
\partial_{a_\nu} \Phi = -\frac{A_\nu}{z - a_\nu} \Phi, \quad A_\nu \in M_N(\mathbb{C})
$$

The isomonodromic $\tau$-function is defined by the closed 1-form

$$
d \ln \tau = \sum_{\mu < \nu} \text{tr}(A_\mu A_\nu) \, d \ln (a_\mu - a_\nu).
$$
3-point Correlator from Tau

- Consider the case $n = 3$. Since $A_1 + A_2 + A_3 = 0$, coefficients $\text{Tr} \ A_\mu A_\nu$ are conserved quantities.
- Thus we can integrate the $\tau$-function to

$$
\tau(a_1, a_2, a_3) = \text{const} \ (a_1 - a_2)^{\Delta_3 - \Delta_2 - \Delta_1} (a_1 - a_3)^{\Delta_2 - \Delta_1 - \Delta_3} (a_2 - a_3)^{\Delta_1 - \Delta_2 - \Delta_3},
$$

where

$$
\Delta_\nu = \frac{1}{2} \text{Tr} \ A_\nu^2
$$

- We can also show that $\tau(a)$ obeys global Ward identities.
Correlators From Tau

- Define

\[ \mathcal{J}(z) = \Phi^{-1} \partial_z \Phi = \Phi^{-1} A(z) \Phi \]

then

\[ \partial_{a \mu} \ln \tau = \frac{1}{2} \operatorname{res}_{z=a\mu} \operatorname{Tr} \mathcal{J}^2(z) \]

and

\[ \Phi(z \to z_0) = 1^N + \mathcal{J}(z_0)(z - z_0) + (\mathcal{J}^2(z_0) + \partial \mathcal{J}(z_0)) \frac{(z - z_0)^2}{2} + \ldots \]

- We suppose then that

\[ \Phi_{jk}(z) = (z - z_0)^{2\Delta} \frac{\langle \mathcal{O}_{L_1}(a_1) \ldots \mathcal{O}_{L_n}(a_n) \bar{\varphi}_j(z_0) \varphi_k(z) \rangle}{\langle \mathcal{O}_{L_1}(a_1) \ldots \mathcal{O}_{L_n}(a_n) \rangle} \]

with primary fields in a 2D CFT with central charge \( c \)
Primary Fields, Descendants and Matrix Elements

- Primary fields $\mathcal{O}_\Delta(0)|0\rangle = |\Delta\rangle$ correspond to highest-weight modules.
- Descendant states $L_{-\lambda}\mathcal{O} = L_{-\lambda_N} \ldots L_{-\lambda_1}\mathcal{O}$ are identified with Young diagrams.
- Ward identities allow us to calculate the matrix elements with one descendant in terms of structure constant.

$$
\langle \mathcal{O}_3(\infty) \mathcal{O}_2(t) L_{-\lambda} \mathcal{O}_1(0) \rangle = C(\Delta_3, \Delta_2, \Delta_1) \gamma_\lambda(\Delta_1, \Delta_2, \Delta_3) t^{\Delta_3-\Delta_1-\Delta_2-|\lambda|}
$$

$$
\gamma_\lambda(\Delta_1, \Delta_2, \Delta_3) = \prod_{j=1}^{N} \left( \Delta_1 - \Delta_3 + \lambda_j \Delta_2 + \sum_{k=1}^{j-1} \lambda_k \right)
$$
OPE of Two Primary Fields

\[ O_2(t)O_1(0) = \sum_{\alpha} \sum_{\mu \in Y} C(\Delta_\alpha, \Delta_2, \Delta_1) \beta_\mu(\Delta_\alpha, \Delta_2, \Delta_1) t^{\Delta_\alpha - \Delta_1 - \Delta_2 + |\mu|} L_{-\mu} O_\alpha(0) \]

\[ \beta_\lambda(\Delta_\alpha, \Delta_2, \Delta_1) = \sum_{\mu \in Y} [Q(\Delta_\alpha)]^{-1} \lambda_\mu \gamma_\mu(\Delta_\alpha, \Delta_2, \Delta_1) \]

We use this result to write

\[ \bar{\phi}_j(z_0) \phi_k(z) = (z - z_0)^{-2\Delta} \left[ \delta_{jk} + J_{jk}(z_0)(z - z_0) + \left( \frac{4\Delta}{c} T(z_0) \delta_{jk} + (\partial J_{jk})(z_0) + S_{jk}(z_0) \right) \frac{(z - z_0)^2}{2} + O((z - z_0)^3) \right] \]
Tau function as Correlator

By comparing the above result with the Taylor expansion of $\Phi(z \to z_0)$

$$\text{tr} \mathcal{J}^2(z) = \frac{\langle \mathcal{O}_{L_1}(a_1) \ldots \mathcal{O}_{L_n}(a_n) T(z) \rangle}{\langle \mathcal{O}_{L_1}(a_1) \ldots \mathcal{O}_{L_n}(a_n) \rangle} \frac{4N\Delta}{c}$$

The OPE of $T(z)$ with primaries give

$$\frac{\langle \mathcal{O}_{L_1}(a_1) \ldots \mathcal{O}_{L_n}(a_n) T(z) \rangle}{\langle \mathcal{O}_{L_1}(a_1) \ldots \mathcal{O}_{L_n}(a_n) \rangle} = \sum_{\nu=1}^{n} \left\{ \frac{\tilde{\Delta}_\nu}{(z - a_\nu)^2} + \frac{1}{z - a_\nu} \partial_{a_\nu} \ln \langle \mathcal{O}_{L_1}(a_1) \ldots \mathcal{O}_{L_n}(a_n) \rangle \right\}$$
Taking the residue, we finally get

\[ \tau(a) = \langle \mathcal{O}_{L_1}(a_1) \cdots \mathcal{O}_{L_n}(a_n) \rangle^{\frac{2N\Delta}{c}} \]

and \( \Delta_\nu = \frac{1}{2} \text{Tr} \ A^2_\nu = \theta^2_\nu \)

We restrict to the case \( c = 2N\Delta \). Realized by a CFT with \( N \) free complex fermions with \( \Delta = 1/2 \)

This is similar to \( u(1) \oplus su(N)_1 \) WZW. The \( u(1) \) factor can be factorized by making \( \text{Tr} \ A_\nu = 0 \), which gives then \( c = N - 1 \)

For \( N = 2 \) and \( n = 4 \), we get \( c = 1 \) CFT description of Painlevé VI \( \tau \)-function
Generic 4-point CFT correlator

\[
\langle O_4(\infty) O_3(1) O_2(t) O_1(0) \rangle = \sum_{\alpha} C(\Delta_4, \Delta_3, \Delta_{\alpha}) C(\Delta_{\alpha}, \Delta_2, \Delta_1) t^{\Delta_{\alpha} - \Delta_1 - \Delta_2} \\
\times B_c(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_{\alpha}; t)
\]

where the conformal blocks are model independent

\[
B_c(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t) = \sum_{\lambda, \mu \in \mathcal{Y}} \gamma_{\lambda}(\Delta, \Delta_3, \Delta_4) \left[ Q(\Delta) \right]_{\lambda \mu}^{-1} \gamma_{\mu}(\Delta, \Delta_2, \Delta_1) t^{|\lambda|}
\]
Painlevé VI Tau-function via AGT Combinatorics

- AGT correspondence (Alday, Gaiotto and Tachikawa 09’) relates instanton partition function in $\mathcal{N} = 2$ 4D SUSY quiver gauge theories to Liouville conformal blocks
- AGT gives combinatorial expansion of conformal blocks

Painlevé VI $\tau$-function combinatorial expansion (Gamayun, Iorgov and Lisovyy ’12)

$$\tau_{VI}(t) = \sum_{n \in \mathbb{Z}} C_{VI}(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma + n) s_{VI}^n t^{(\sigma+n)2-\theta_0^2-\theta_t^2} \mathcal{B}_{VI}(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma + n; t)$$
AGT Conformal Blocks

\[ \mathcal{B}_{VI}(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma; t) = (1-t)^2 \theta_t \theta_1 \sum_{\lambda, \mu \in Y} \mathcal{B}^{(VI)}_{\lambda, \mu}(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma) t^{\lambda + \mu} \]

\[ \mathcal{B}^{(VI)}_{\lambda, \mu}(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma) = \prod_{(i, j) \in \lambda} \frac{\left( \left( \theta_t + \sigma + i - j \right)^2 - \theta_0^2 \right) \left( \left( \theta_1 + \sigma + i - j \right)^2 - \theta_\infty^2 \right)}{h_{\lambda}(i, j) \left( \lambda'_j + \mu_i - i - j + 1 + 2\sigma \right)^2} \times \prod_{(i, j) \in \mu} \frac{\left( \left( \theta_t - \sigma + i - j \right)^2 - \theta_0^2 \right) \left( \left( \theta_1 - \sigma + i - j \right)^2 - \theta_\infty^2 \right)}{h_{\mu}(i, j) \left( \lambda_i + \mu'_j - i - j + 1 - 2\sigma \right)^2}. \]
Product of Structure Constants

\[ C_{VI}(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma) = \prod_{\epsilon, \epsilon' = \pm} \frac{G(1 + \theta_t + \epsilon \theta_0 + \epsilon' \sigma) G(1 + \theta_1 + \epsilon \theta_\infty + \epsilon' \sigma)}{G(1 + 2 \epsilon \sigma)} \]

where \( G(z + 1) = \Gamma(z)G(z) \) is the Barnes function

- This conjecture has passed numerical testing
## Contents

1. Black Holes, CFT and Isomonodromy
   - Black Hole Entropy via CFT
   - Scattering from Monodromies
   - Isomonodromy in $D = 4$ Kerr-de Sitter

2. Isomonodromy and CFT
   - Isomonodromic $\tau$-function and $c = 1$ CFT
   - Quantization of Isomonodromic Deformations
Canonical Quantization of Isomonodromic System

- Heun equation can be rewritten as

\[ H(z, b^2 \partial_z, z_i) y(z) = \left( b^4 \partial_z^2 + \sum_{i=1}^{4} \left( \frac{\delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right) \right) y(z) = 0, \]

where \( H(\lambda, \mu, z_i) \) is related to the isomonodromic hamiltonian.

- If we make the canonical quantization

\[ \hat{c}_i \Psi(z, z_i) = b^2 \partial_{z_i} \Psi(z, z_i), \quad \Delta_i = \delta_i / b^2, \]

the wave equation is given by the 5-point BPZ equation

\[ \left( b^2 \partial_z^2 + \sum_{i=1}^{4} \left( \frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \partial_{z_i} \right) \right) \Psi(z, \{z_i\}) = 0 \]
Classical Limit

- In the classical limit $b^2 \to 0 \ (c \to \infty)$, the conformal block exponentiate to $f_\sigma(t)$ and gives
  
  $$c_t = \partial_t f_\sigma(t)$$

  for generic 4-point block (Litvinov, Lukyanov, Nekrasov and Zamolodchikov ’13)
- Suggests that $f_\sigma(t)$ is related to the $\tau$-function
- Connection with AGT?
Conclusions

- String theory and Kerr/CFT suggest a CFT description of BH microstates
- Isomonodromic deformations naturally appear in linear perturbations of non-extremal BHs and give a procedure to calculate quasinormal modes
- Isomonodromic $\tau$-function is a correlator of monodromy primaries in $c=1$ CFT and its canonical quantization gives BPZ equation in Liouville
- Painlevé VI also appears in $c \to \infty$ limit
- The $\tau$-function also seems connected to the AGT relation
Questions

- Is there any deeper reason for this connection between $c = 1$ and $c = \infty$ limits?
- Does this classical-quantum integrability leads to new hints on the BH sector of quantum gravity?
Questions

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Thanks!
Relation with Fuchsian Equation

- Fuchsian ODE normal form with $n$ finite singular points

\[ \psi''(z) + T(z)\psi(z) = 0, \quad T(z) = \sum_{i=1}^{n} \left( \frac{\delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right), \]

\[ \sum_{i=1}^{n} c_i = 0, \quad \sum_{i=1}^{n} (c_i z_i + \delta_i) = 0, \quad \sum_{i=1}^{n} (c_i z_i^2 + 2\delta_i z_i) = 0 \]

- Local monodromies: $\delta_i = (1 - \theta_i^2)/4$

- Accessory parameters $c_i$ have global properties

- $2(n - 3)$ independent parameters: $(c_i, z_i)$
Symplectic Structure of Flat $\text{SL}(2, \mathbb{C})$ Connections

- Moduli space of flat connections $A \sim$ moduli space of monodromy group
- Atiyah-Bott symplectic structure

$$\Omega = \sum_{i=1}^{n-3} dc_i \wedge dz_i = \sum_{i=1}^{n-3} d\nu_i \wedge d\mu_i$$

where $(\nu_i, \mu_i)$ are trace coordinates (Nekrasov et al 2011)
- Canonical transformation connects both set of coordinates
- Suggests analytical approach to find composite monodromies
- Relation with classical conformal blocks of 2D CFT
Using that $\det A_i^0 = -\theta_i^2/4$ and $\det \Lambda = -\sigma_{0t}^2/4$

\[
\Lambda + \frac{1}{2} \sigma \mathbb{1} = \frac{1}{4\theta_\infty} \begin{pmatrix}
(-\theta_\infty - \theta_1 + \sigma)(\theta_\infty - \theta_1 - \sigma) & (-\theta_\infty - \theta_1 + \sigma)(\theta_\infty + \theta_1 + \sigma) \\
(\theta_\infty - \theta_1 + \sigma)(\theta_\infty - \theta_1 - \sigma) & (\theta_\infty - \theta_1 + \sigma)(\theta_\infty + \theta_1 + \sigma)
\end{pmatrix}
\]

\[
A_1^0 + \frac{1}{2} \theta_1 \mathbb{1} = \frac{1}{4\theta_\infty} \begin{pmatrix}
-(\theta_\infty - \theta_1)^2 + \sigma^2 & (\theta_\infty + \theta_1)^2 - \sigma^2 \\
-(\theta_\infty - \theta_1)^2 + \sigma^2 & (\theta_\infty + \theta_1)^2 - \sigma^2
\end{pmatrix}
\]

\[
A_0^0 + \frac{1}{2} \theta_0 \mathbb{1} = G_1 \frac{1}{4\sigma} \begin{pmatrix}
(\theta_0 - \theta_t + \sigma)(\theta_0 + \theta_t + \sigma) & (\theta_0 - \theta_t + \sigma)(-\theta_0 - \theta_t + \sigma) \\
(\theta_0 - \theta_t - \sigma)(\theta_0 + \theta_t + \sigma) & (\theta_0 - \theta_t - \sigma)(-\theta_0 - \theta_t + \sigma)
\end{pmatrix} G_1^{-1}
\]

\[
A_t^0 + \frac{1}{2} \theta_t \mathbb{1} = G_1 \frac{1}{4\sigma} \begin{pmatrix}
(\theta_t + \sigma)^2 - \theta_0 & -(\theta_t - \sigma)^2 + \theta_0^2 \\
(\theta_t + \sigma)^2 - \theta_0 & -(\theta_t - \sigma)^2 + \theta_0^2
\end{pmatrix} G_1^{-1}.
\]