Non-Abelian anyons with non-Abelian gauge potentials?

The Laughlin ground-state has abelian excitations: the Berry phase is a *number*.

How we can make it a *matrix*?

First (poor man) guess: add a degree of freedom, i.e. consider a two-component Bose gas, and put a non-Abelian gauge potential

Well, not so easy…

Single-particle Hamiltonians

$$H = \left(p_x + A_x\right)^2 + \left(p_y + A_y\right)^2$$

For a single component with only a magnetic field along z:

$$A_x = 0;\ A_y = Bx\ \left(Landau\ gauge\right)$$

$$or$$

$$A_x = -\frac{B}{2}y;\ A_y = \frac{B}{2}x\ \left(symmetric\ gauge\right)$$

Using two-component gases & rotations and/or tripod schemes it is possible to have A_x, A_y **2x2 matrices with**

$$\left[A_x, A_y\right] \neq 0$$

$$\Uparrow$$

Non-Abelian gauge potentials

2D atoms in a non-Abelian magnetic field: An example

$$H=\left(p_x+A_x\right)^2+\left(p_y+A_y\right)^2$$

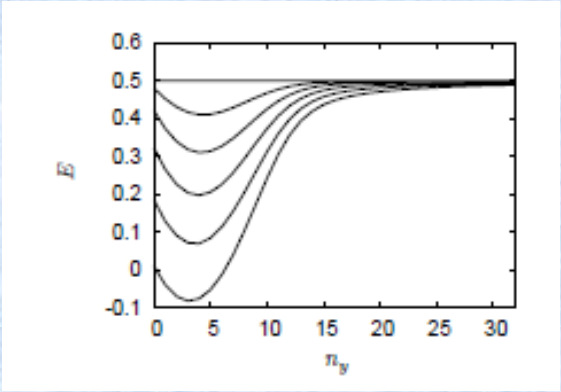
An example:

$$A_x=q\,M_x\;;\;A_y=B\,M_y\,x$$

2x2 matrices

with

$$M_y=\sigma_z;\;M_x=\sigma_y$$



Degeneracy of Landau levels is in general broken

[A. Jacob et al., New J. Phys. (2008)]

2D atoms in a symmetric non-Abelian magnetic field: single particle (I)

$$H=\left(p_x+A_x\right)^2+\left(p_y+A_y\right)^2$$

A choice:

$$\boxed{A_x=q\sigma_x-\frac{B}{2}y\ ;\ A_y=q\sigma_y+\frac{B}{2}x}$$

Reasons:

1) degeneracy of Landau levels <u>not</u> broken

2) analytical single particle energy levels

[Y.A. Bychkov and E.I. Rashba, J. Phys. C (1984)]

3) realistically implementable

[M. Burrello and A. Trombettoni, PRL (2010); PRA (2011)]

2D atoms in a symmetric non-Abelian magnetic field: single particle (II)

$$H=\left(p_x+A_x\right)^2+\left(p_y+A_y\right)^2\equiv H_{abel.}+H_{non-abel.}$$

$$A_x=q\sigma_x-\frac{B}{2}y\ ;\ A_y=q\sigma_y+\frac{B}{2}x$$

One finds

$$H_{abel.}=2q^2+B+\frac{1}{4}d^{+d}\quad\left(d^+=B\bar{z}-4\partial/\partial z\,;\ z=x+iy\right)$$

$$H_{non-abel.}=q\begin{pmatrix}0 & id\\ -id^+ & 0\end{pmatrix}$$

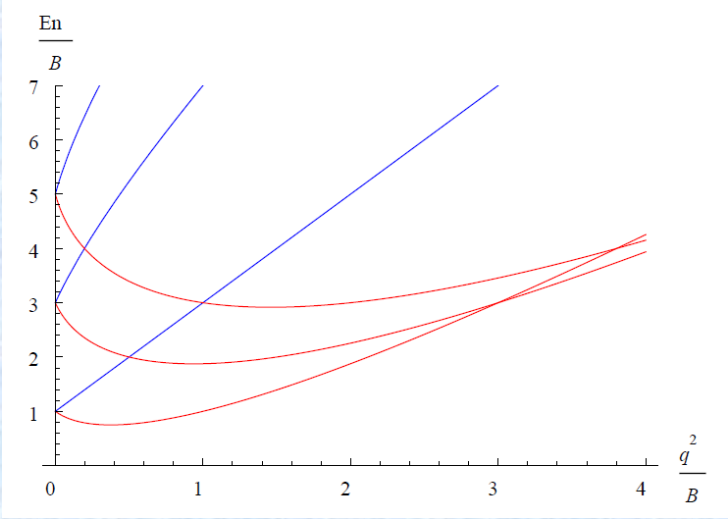
$$\boxed{\epsilon_n^{\pm}=2Bn+2q^2\pm\sqrt{B^2+8q^2Bn}}$$

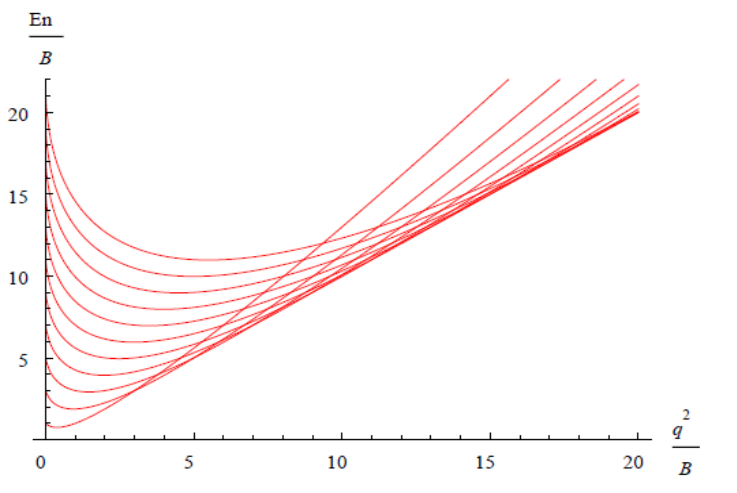
Degeneracy of Landau levels is preserved, although each level splits in two due to non-abelian term q

2D atoms in a symmetric non-Abelian magnetic field: single particle (III)

$$H=\left(p_x+A_x\right)^2+\left(p_y+A_y\right)^2=H_{abel.}+H_{non-abel.}$$

$$A_x=q\sigma_x-\frac{B}{2}y\ ;\ A_y=q\sigma_y+\frac{B}{2}x$$





Landau levels split in two due to non-abelian term q

2D atoms in non-Abelian magnetic fields: Adding the interactions

q=0 ➡ usual ("Abelian") case with lowest Landau levels; (strong) interaction gives the Laughlin state

q finite ➡ *deformed* Laughlin state

$$\Psi^m = \prod_j G_j \Psi_L^m; \quad \Psi_L^m = \prod_{i<j} (z_i - z_j)^m e^{\frac{-B}{4}\sum_i |z_i|^2} |\downarrow\downarrow\downarrow\ldots\downarrow\rangle$$

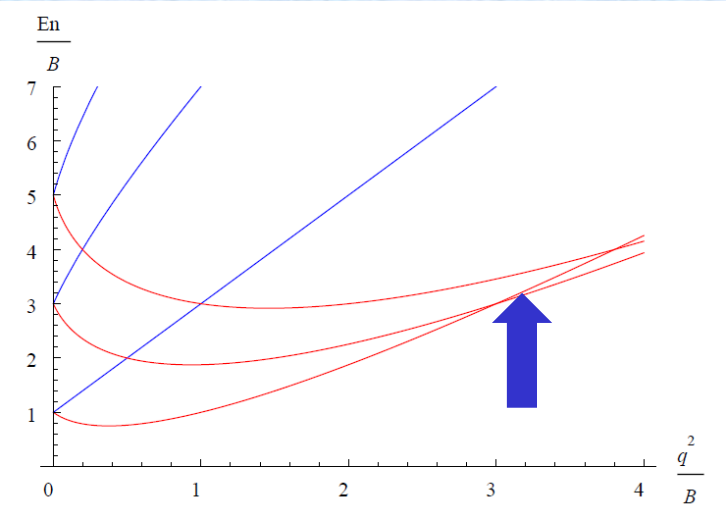
where

$$G \equiv c_\uparrow \sigma_x + c_\downarrow d^+; \; d = Bz - 4\partial/\partial z$$

$$c_\uparrow = B + 2q\sqrt{2B} + \sqrt{B^2 + 8q^2 B}; \; c_\downarrow = \frac{i}{2\sqrt{2B}}\left(B - 2q\sqrt{2B} - \sqrt{B^2 + 8q^2 B}\right)$$

...but excitations are <u>Abelian</u>...

Non-Abelian excitations at the degeneracy points



At the degeneracy points, ground-states with <u>non-Abelian</u> excitations are found: a *deformed* Moore-Read

$$\Psi_{MR} = S\left(\prod_i G_{1,i} \prod_j G_{2,j}\right) Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2 \, e^{\frac{-B}{4}\sum_i |z_i|^2} |\downarrow\downarrow \ldots \downarrow\rangle$$

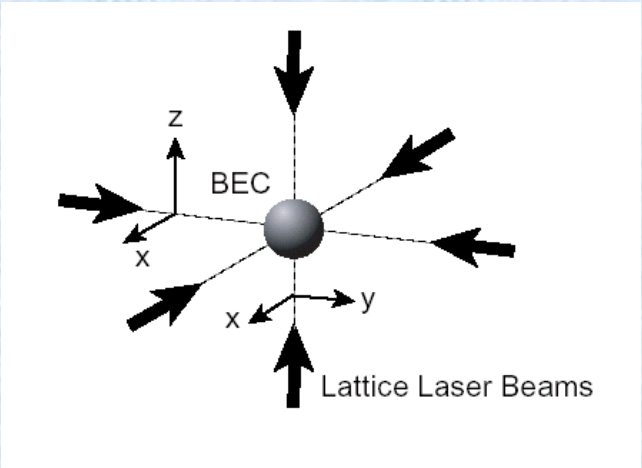
[M. Burrello and A. Trombettoni, PRL (2010)]

Quantum Simulations of Field Theories

↓

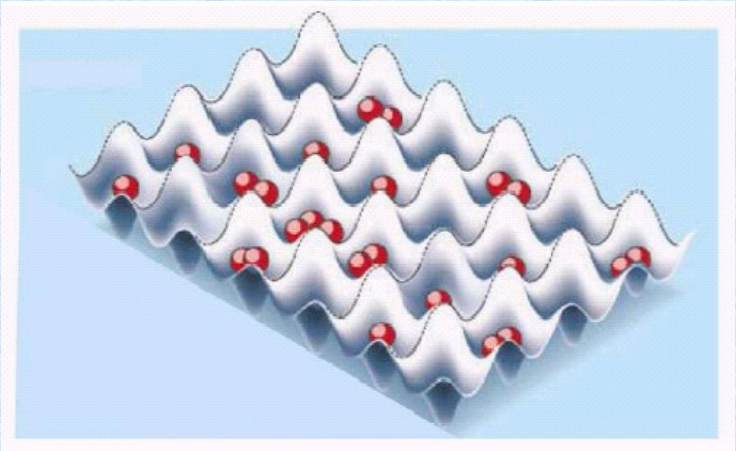
Realization by purpose of field theories of interest as effective low-energy models in an experimental setup with highly tunable parameters

Ultracold bosons in an optical lattice



$$V_{opt}(x) = V_0 \sin^2(kx)$$

e.g., a 1D lattice



It is possible to control:
- barrier height
- interaction term
- the shape of the network
- the dimensionality (1D, 2D, …)
- the tunneling among planes or among tubes (in order to have a layered structure)
…

Effective Hamiltonian for ultracold fermions in optical lattices

Similarly, for a <u>dilute</u> single-species Fermi gas the effective Hamiltonian is

$$\hat{H} = -t \sum_{<i,j>} \left(\hat{c}_i^+ \hat{c}_j + h.c. \right) \equiv -t \sum_{i,j} A_{ij} \hat{c}_i^+ \hat{c}_j$$

$$\left(f \leq 1 \right)$$

TIGHT-BINDING HAMILTONIAN

Notice that informations about the geometry and the Wannier functions are into the matrix A and the coefficients t, U:

$$t = -\int d\vec{r}\, \Phi_i^*\left(\vec{r}\right) \left[\frac{-\hbar^2}{2\mathrm{m}} \nabla^2 + V_{opt}\left(\vec{r}\right) \right] \Phi_j\left(\vec{r}\right)$$

$$U = g_0 \int d\vec{r}\, \left| \Phi_i\left(\vec{r}\right) \right|^4$$

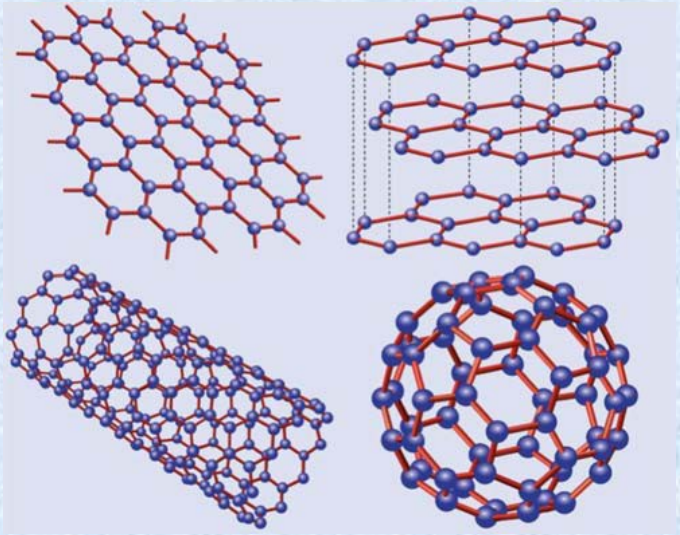
Graphene physics: a very short reminder

Graphene

Graphite

Carbon nanotubes

Fullerene



Graphene → honeycomb lattice of carbon atoms:
π band is half-filled
lattice spacing 1.42 Angstroms
t~2.8 eV
$v_F \sim 10^6$ m/s

[A.H. Castro-Neto et al., Rev. Mod. Phys. (2009)]

The needed “ingredients” for the simulation of graphene with ultracold atoms...

- **Bosons and/or fermions** ⬅
- **Geometry (1D / 2D)**
- **Long-range interactions**
- **Disorder**
- **Time- and space- dependence of the parameters of the Hamiltonian**
- **Explicit tuning of the interactions via Feshbach resonances**
- **Simulate a magnetic field through a rotation or with optical tools**
- **Optical lattices (i.e., periodic potentials with minima of the potential located on a lattice)** ⬅

Quantum simulation of graphene properties (I)

Implementable putting ultracold fermions in lattices having Dirac points

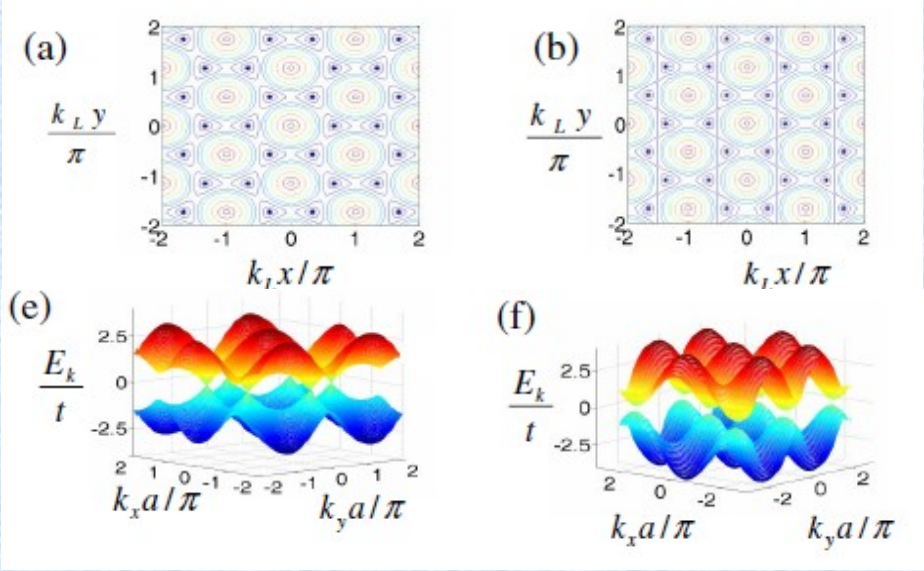
e.g., in 2D, using the honeycomb lattice itself: using three optical lattices

$$V(x, y) = \sum_{j=1,2,3} V_j \sin^2[k_L(x\cos\theta_j + y\sin\theta_j) + \pi/2]$$

$$\theta_1 = \pi/3, \theta_2 = 2\pi/3, \theta_3 = 0$$

Tight-binding model on the honeycomb (alias, graphene)

Quantum simulation of graphene properties (II)



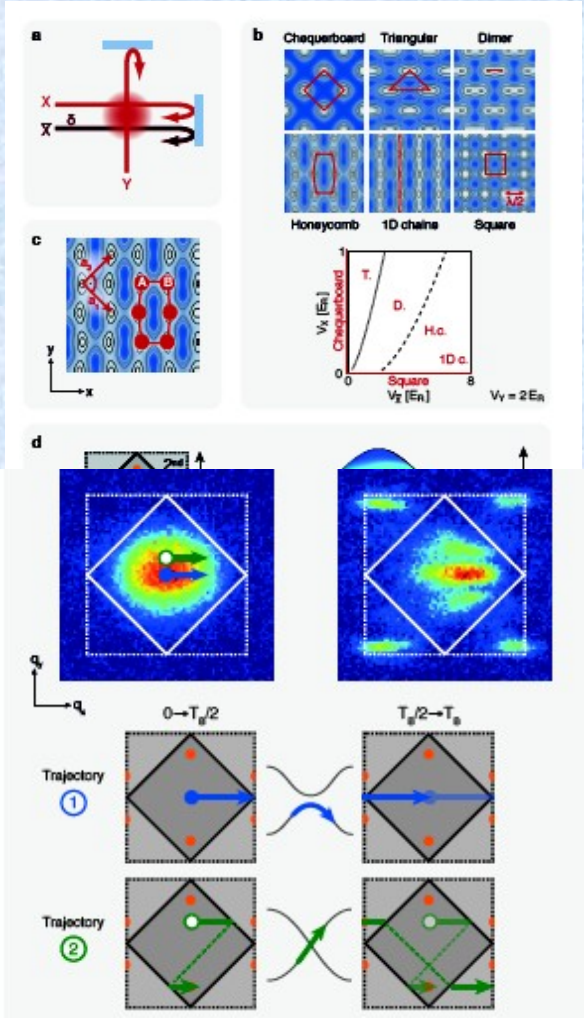
$$E_{\mathbf{k}} = \pm t\sqrt{2 + \beta^2 + 2\cos(k_y a) + 4\beta\cos(\sqrt{3}k_x a/2)\cos(k_y a/2)}.$$

$$t_1 = t_2 = t \text{ and } t_3 = \beta t$$

[from S.L. Zhu, B. Wang, and L.-M. Duan, PRL (2007)]

Experimental realization

$$V(x,y) = -V_{\overline{X}} \cos^2(kx + \theta/2) - V_X \cos^2(kx)$$
$$-V_Y \cos^2(ky) - 2\alpha\sqrt{V_X V_Y} \cos(kx)\cos(ky)\cos\varphi$$



[L. Tarruell et al., Nature (2012)]

The 3D case

Not a straigthforward generalization of the 2D case: indeed, having 2D honeycomb coupled along the z-direction destroys in general the Dirac cones.

More formally:

$$\hat{H} = -t \sum_{i,j} A_{ij} \hat{c}_i^+ \hat{c}_j$$

adjacency matrix of the graph [cfr. N. Biggs, Algebraic Graph Theory]

$$-t \sum_j A_{ij} \varphi_\alpha(j) = \epsilon_\alpha \varphi_\alpha(i) \qquad \hat{d}_\alpha = \sum_{\vec{j}} \varphi_\alpha(j) \hat{c}_{\vec{j}}$$

$$\hat{H} = \sum_\alpha \epsilon_\alpha \hat{d}_\alpha^+ \hat{d}_\alpha$$

The requests are that:

i) the single particle spectrum has Dirac points (and cones) and the graph has spectral dimension 3

ii) that the the adjacency matrix has nearest-neighbour couplings

iii) not too many lasers are needed...

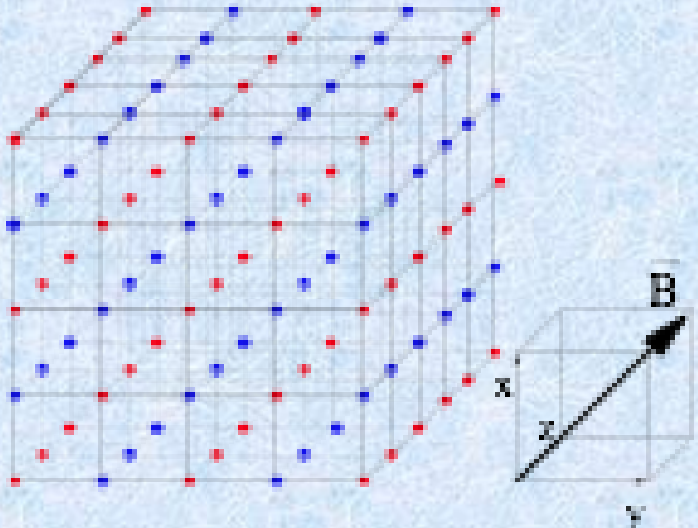
Although symmetries have been studied *[A.A. Abrikosov and S.D. Beneslavskii, JETP (1970) – J.L. Manes, PRB (2012)]***, not easy to satisfy in practice i)-ii)-iii)...**

For our purposes: single-species Fermi gas in a π-flux magnetic field (at half filling)

$$\hat{H} = -t \sum_{<i,j>} \hat{c}_i^{+} e^{-i a_{ij}} \hat{c}_j + h.c.$$

$$a_{ij} = \int_i^j \vec{A} \cdot d\vec{l} \qquad \vec{B} \equiv rot\, \vec{A} = \pi(1,1,1)$$

(we can also assume different hoppings t_x, t_y and t_z along the three directions x, y and z)



Single-particle spectrum and Dirac cones (I)

Using the Hasegawa's gauge:

$$\vec{A}=\pi\left(0, x-y, y-x\right)$$

[Y. Hasegawa, J. Ph.Soc. Jap. (1990)]

one gets

$$E_{\vec{k}}=\pm 2\sqrt{t_x^2\cos^2 k_x+t_y^2\cos^2 k_y+t_z^2\cos^2 k_z}$$

with k belonging to the first (magnetic) Brillouin zone.

[L.Lepori, G. Mussardo, and A. Trombettoni, Europhys. Lett. (2010);
see also the recent experimental proposal by T. Dubcek et al, PRL (2015)]

Single-particle spectrum and Dirac cones (II)

- **For t_z =0 the results for the 2D case with π-flux are retrieved** [I. Affleck and].B. Marston, PRB (1988)] **are retrieved.**

- **Excitations around the two inequivalent Dirac points obey the 3D Dirac equation.**

- **In the limit of vanishing t_z one retrieves the 2D Dirac equation.**

- **A mass term can be added using a Bragg pulse.**

- **The Dirac points does not depend on t_x, t_y and t_z.**

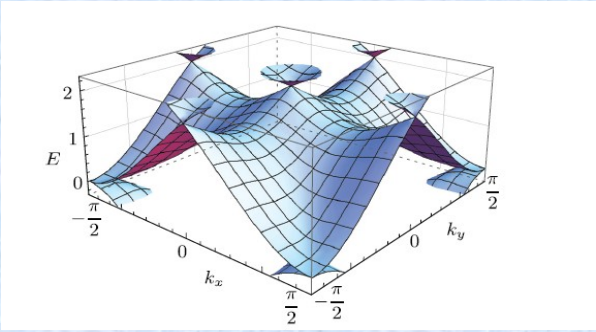
- **With a spatial control of the synthetic magnetic field one can simulate e.m. coupling.**

[L.Lepori, G. Mussardo, and A. Trombettoni, Europhys. Lett. (2010);
G. Mazzucchi, L. Lepori, and A. Trombettoni, J. Phys. B (2013)]

Adding a non-Abelian term (I)

$$A_x = q\sigma_x - \frac{B}{2} y \; ; \; A_y = q\sigma_y + \frac{B}{2} x$$

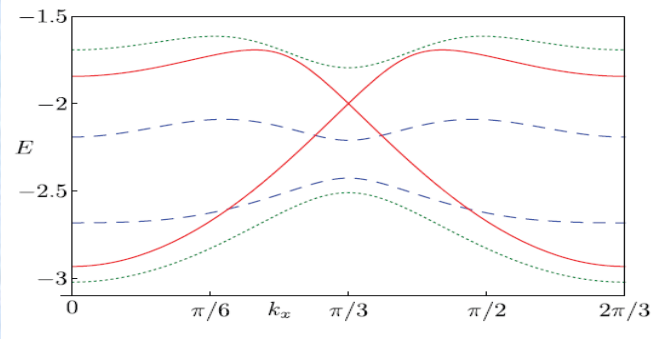
Flux π of the Abelian magnetic potential B → Dirac cones, but no non-trivial topological phases with symmetry protected edge states



Adding a non-Abelian term (II)

$$A_x = q\sigma_x - \frac{B}{2} y \; ; \; A_y = q\sigma_y + \frac{B}{2} x$$

Flux 2π/3 of the Abelian magnetic potential B → time-reversal symmetry is broken, gap open, one has symmetry-protected edge modes and topological phase transitions between them (varying q)

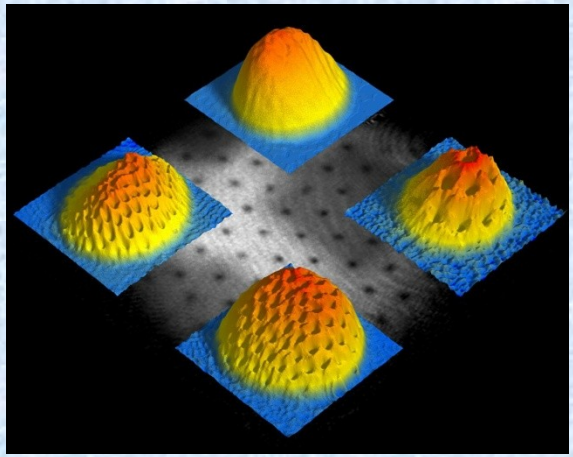


two lowest-energy states for different values of q
[M. Burrello, I. C. Fulga, E. Alba, L. Lepori, and A. Trombettoni, PRA (2013)]

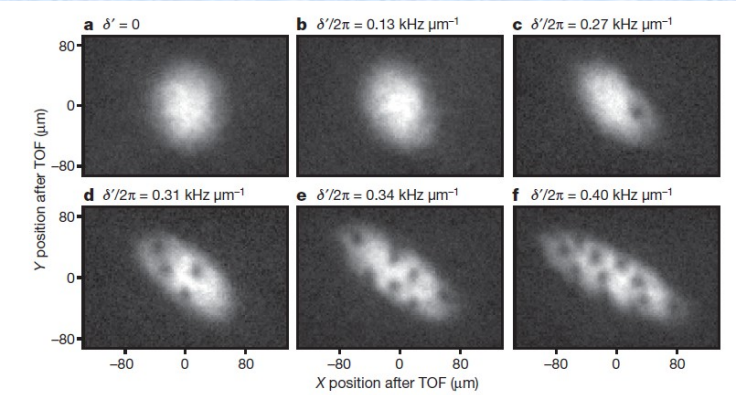
PT symmetry does not have to be necessarily broken to have Weyl semimetals! [L. Lepori, I. C. Fulga, A. Trombettoni and M. Burrello, arXiv: 1506.04761]

Simulating a synthetic magnetic field:

- **Using rotating traps**



- **With spatially dependent optical couplings between internal states of the atoms** [Y.-J. Lin et al., Nature (2009)]



Laughlin ground-states for 2D gases in rotation

Effect of a <u>strong</u> interaction:

$$H=\sum_i H_i+g_{2D}\hbar\omega\sum_{i<j}\delta(z_i-z_j)$$

For $g_{2D}\rightarrow\infty$

$$\varphi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)\bar{\varphi}(z_1,\ldots,z_N)\rightarrow(\omega-\Omega)L_z\bar{\varphi}=E\bar{\varphi}$$

$$\bar{\varphi}=det\begin{pmatrix}1 & z_1 & z_1^2 & \ldots & z_1^N\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 1 & z_N & z_N^2 & \ldots & z_N^N\end{pmatrix}=\prod_{i<j}(z_i-z_j)$$

$$\boxed{\psi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^2 e^{-\sum_i |z_i|^2/4l^2}}$$

Laughlin wavefunction

Abelian excitations

Excitations of the Laughlin ground-state:

$$\psi_{R_0}(z_1,\ldots,z_N)=\prod_i (z_i-R_0)\prod_{i<j}(z_i-z_j)^2 e^{-\sum_i |z_i|^2/4l^2}$$

For ultracold atomic systems, an <u>hole</u> in R_0 can be created with a laser centered in R_0; two excitations (say in R_0 and R_1) can be created

or

one excitation can be moved - i.e., $R_0=R_0(time)$ – around another ➡ a $\pi/2$ phase is acquired

Non - Abelian excitations

[Moore and Read (1991)]

In a non-abelian quantum Hall state quasi-particles obey non-abelian statistics, meaning that (for example):

with 2N quasi-particles at fixed positions, the ground state is 2^N-degenerate

the interchange of quasi-particles shifts between ground states (i.e., permutations between quasi-particles positions unitary transformations in the ground state subspace)

An example: *Moore-Read states*

$$H=\sum_i \sum_{j>i} \sum_{k>j} \delta(z_i - z_j)\delta(z_j - z_k)$$

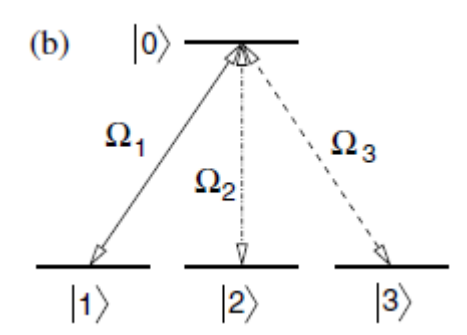
$$\Downarrow$$

$$\psi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^2 \prod_i e^{-|z_i|^2/4l^2} \cdot Pf\left(\frac{1}{z_i-z_j}\right)$$

Rotation & artificial magnetic fields (I)

Rotation for neutral atoms ~ **Magnetic field for charged particles**

Another tool for simulate magnetic fields for a two-component Bose gas is purely optical: Rabi pulses couples two states with two dark states (tripod structure)



[see the review J. Dalibard, F. Gerbier, G. Juzelunas, and P. Ohberg, RMP (2011)]

Main available “ingredients”

- Bosons and/or fermions
- Geometry (1D / 2D) ⬅
- Long-range interactions
- Add disorder
- Time-dependence (and to a certain extent space-dependence) of the parameters of the Hamiltonian
- Explicit tuning of the interactions via Feshbach resonances
- Simulate a magnetic field through a rotation or with optical tools ⬅
- Optical lattices (i.e., periodic potentials having minima located on a lattice)
- …

Rotation & artificial magnetic fields (III)

Other promising schemes:

-optical flux lattices [N. Cooper, PRL (2011)]

-generating a geometric phase with specific arrangements on lattices using two stable states [K. Osterloh et al., PRL (2005); M. Aidelsburger et al., PRL (2011)]

⬇

for this purpose Yb atoms can be useful [F. Gerbier and J. Dalibard, New J. Phys. (2010)]: two electronically-excited metastable states – 7 isotopic stable forms [5 bosons & 2 fermions] – selective trapping of the two states

Rotation & artificial magnetic fields (II)

$$\Omega_1 = \Omega \sin(\theta)\cos(\varphi)\, e^{i S_1}$$
$$\Omega_2 = \Omega \sin(\theta)\sin(\varphi)\, e^{i S_2}$$
$$\Omega_3 = \Omega \cos(\theta)\, e^{i S_3}$$

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\nabla - A\right)^2 + V\right]\Psi}$$

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

A is a 2x2 matrix:

$$A_{11} = \hbar\left(\cos^2(\varphi)\nabla S_{23} + \sin^2(\varphi)\nabla S_{13}\right)$$
$$A_{12} = \hbar\cos(\theta)\left(\frac{1}{2}\sin(2\varphi)\nabla S_{12} - i\nabla\varphi\right)$$
$$A_{22} = \hbar\cos^2(\theta)\left(\cos^2(\varphi)\nabla S_{23} + \sin^2(\varphi)\nabla S_{23}\right)$$

Diagonalization of the Hamiltonian

Let us consider now the non–abelian part of the Hamiltonian. The eigenvalues of H_{na} are $\lambda^{\pm} = \pm 2q\sqrt{2B(n+1)}$ and the corresponding eigenstates $\varphi_n^{\pm}$ can be expressed in terms of the eigenstates of H_a:

$$\varphi_n^{\pm} = \psi_{n-1}\left|\uparrow\right\rangle \pm \psi_n\left|\downarrow\right\rangle \tag{10}$$

$$H\varphi_n^{\pm} = \left(2q^2 + 2Bn \pm 2q\sqrt{2Bn}\right)\varphi_n^{\pm} - B\varphi_n^{\mp}$$

The eigenvalues of H_n are:

$$\varepsilon_n^{\pm} = 2Bn + 2q^2 \pm \sqrt{B^2 + 8q^2Bn}$$

and the unnormalized eigenstates of the Hamiltonian H are:

$$\chi_n^{\pm} = B\varphi_n^{+} + 2q\sqrt{2Bn}\varphi_n^{-} \mp \sqrt{B^2 + 8q^2Bn}\varphi_n^{-}$$

G's operators

$$
\begin{aligned}
c_{\uparrow,n} &= B + 2q\sqrt{2Bn} + \sqrt{B^2 + 8q^2 Bn} \\
c_{\downarrow,n} &= i\frac{B - 2q\sqrt{2Bn} - \sqrt{B^2 + 8q^2 Bn}}{2\sqrt{2Bn}}
\end{aligned}
$$

$$
\mathcal{G}_1 \equiv c_{\uparrow,1}\sigma_x + c_{\downarrow,1}d^\dagger,
$$

$$
\Psi_n^{(m)} = \prod_{j \leq N} \mathcal{G}_{n;j} \Lambda_N^{(m)}
$$

$$
\Lambda_N^{(m)} = \prod_{i<j}^{N} (z_i - z_j)^m e^{-\frac{B}{4}\sum_i^N |z_i|^2} |\downarrow\downarrow \dots \downarrow\rangle
$$

Ground-states at the degeneracy points

$$\Omega_c = \mathcal{A}\left[\prod_{k\in A}\mathcal{G}_{1;k}\prod_{i<j\in A}(z_i - z_j)\prod_{l\in B}\mathcal{G}_{2;l}\prod_{i<j\in B}(z_i - z_j)\right] e^{-\frac{B}{4}\sum_i^{2N}|z_i|^2}\,|\downarrow\downarrow\ldots\downarrow\rangle\,.$$

$$\Omega_{qh} = \mathcal{A}\left[\prod_A \mathcal{G}_{1;k}\prod_B \mathcal{G}_{2;l}\prod_A (z_i - z_j)\prod_A\prod_{h=1}^{M}(z_i - \zeta_h)\right.$$
$$\left.\prod_B (z_i - z_j)\prod_B\prod_{h=M+1}^{2M}(z_i - \zeta_h)\right] e^{-\frac{B}{4}\sum_i^{2N}|z_i|^2}\,|\downarrow\ldots\downarrow\rangle$$

$$\Omega_{\mathrm{Hf}}^{m} = \mathrm{Hf}\left((\mathcal{G}_{1;i}\mathcal{G}_{2;j} - \mathcal{G}_{2;i}\mathcal{G}_{1;j})\frac{1}{z_i - z_j}\right)\prod_{i<j}^{2N}(z_i - z_j)^m\, e^{-\frac{B}{4}\sum_i^{2N}|z_i|^2}\,|\downarrow\downarrow\ldots\downarrow\rangle$$

$$\Omega_{\mathrm{MR}} = \mathcal{S}\left(\prod_{i=1}^{N}\mathcal{G}_{1;i}\prod_{i=1+N}^{2N}\mathcal{G}_{2;i}\right)\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}^{2N}(z_i - z_j)^2\, e^{-\frac{B}{4}\sum_i^{2N}|z_i|^2}\,|\downarrow\downarrow\ldots\downarrow\rangle$$

$$\Omega_{\mathrm{MR}}(\zeta_a,\zeta_b,\zeta_c,\zeta_d) = \mathcal{S}\left(\prod_{i=1}^{N}\mathcal{G}_{1;i}\prod_{i=1+N}^{2N}\mathcal{G}_{2;i}\right)$$
$$\mathrm{Pf}\left(\frac{(z_i - \zeta_a)(z_i - \zeta_b)(z_j - \zeta_c)(z_j - \zeta_d) + i\leftrightarrow j}{z_i - z_j}\right)\Lambda_{2N}^{(2)}$$

Reminder on the Pfaffian

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}$$

$$(Pf\ A)^2 = \det A$$

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}. \quad \mathrm{Pf}(\mathrm{A}) = \sqrt{\det(\mathrm{A})} = \sqrt{0 \cdot 0 - a \cdot (-a)} = a.$$

$$\mathrm{Pf} \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc.$$

$$\mathrm{Pf} \begin{bmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & 0 & \cdots & 0 \\ & & 0 & \lambda_2 & & \\ & 0 & -\lambda_2 & 0 & & 0 \\ \vdots & & & & \ddots & \vdots \\ & 0 & & 0 & \cdots & 0 \quad \lambda_n \\ & & & & & -\lambda_n \quad 0 \end{bmatrix} = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Moore-Read states

$$H = \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \delta'(z_i - z_j)\delta'(z_i - z_k)$$

$$\Psi(z_j) = \prod_{j<k} (z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \cdot \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right)$$

$$\Psi(z_j) = \prod_{j<k} (z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \cdot \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right)$$

$$\Psi_{2\,\mathrm{qh}} = \prod_{j<k} (z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \cdot \mathrm{Pf}\left(\frac{(z_j - \eta_1)(z_k - \eta_2) + (z_j - \eta_2)(z_k - \eta_1)}{z_j - z_k}\right)$$

[see Nayak and Wilczek, cond-mat/9605145]

N.B. **Requiring that for** $H=\left(p_x+A_x\right)^2+\left(p_y+A_y\right)^2$

1) energy spectrum presents a Landau level structure (and the wavefunctions can be expressed as finite sum of terms)

2) each Landau level is degenerate with respect to the angular momentum

3) in the Abelian limit, the Landau levels become degenerate with respect of the spin degree of freedom

only two classes of Hamiltonians are found

$$H=\left(E+h_z\sigma_z\right)d^+d+M_z\sigma_z-iq\sigma_+d+iq\sigma_-d^+$$

$$H=\left(E+h_z\sigma_z\right)d^+d+M_z\sigma_z-iq\sigma_+d^2+iq\sigma_-d^{+2}$$

[M. Burrello and A. Trombettoni, PRA (2011)]

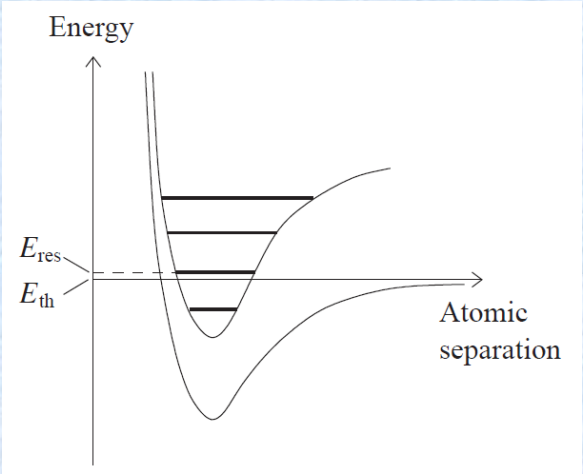
Feshbach resonances

For dilute gases, the interaction between two particles in the same *channel* (i.e., having the same quantum numbers) is

$$V(\vec{x}-\vec{y}) \approx g_0\, \delta(\vec{x}-\vec{y}) \qquad g_0 = \frac{4\pi\hbar^2 a}{m}$$

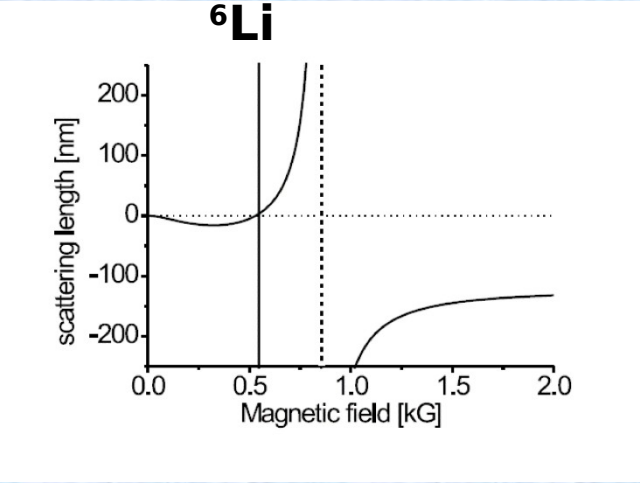
← **s-wave scattering length**

The coupling with other (*closed*) channels modifies the s-wave scattering length:



$$a = a_{background}\left(1 - \frac{C}{B-B_0}\right)$$

e.g., for ^{6}Li



Effective Hamiltonian for ultracold bosons in optical lattices (I)

In second quantization, the full quantum many-body Hamiltonian is

$$\hat{H}=\int d\vec{r}\left(\hat{\psi}^{+}(\vec{r})\left[\frac{-\hbar^{2}}{2\mathrm{m}}\nabla^{2}+V_{opt}(\vec{r})\right]\hat{\psi}(\vec{r})+g_{0}\,\hat{\psi}^{+}(\vec{r})\,\hat{\psi}^{+}(\vec{r})\,\hat{\psi}(\vec{r})\,\hat{\psi}(\vec{r})\right)$$

A very good description of (equilibrium and dynamical) low-energy properties – valid for large values of lattice height - is obtained using the Ansatz

$$\hat{\psi}(\vec{r})=\sum_{i}\hat{b}_{i}\,\Phi_{i}(\vec{r})$$

tight-binding Ansatz
[D. Jaksch et al., PRL (1998)]

Wannier functions
(to be determined)

One gets...

Effective Hamiltonian for ultracold bosons in optical lattices (II)

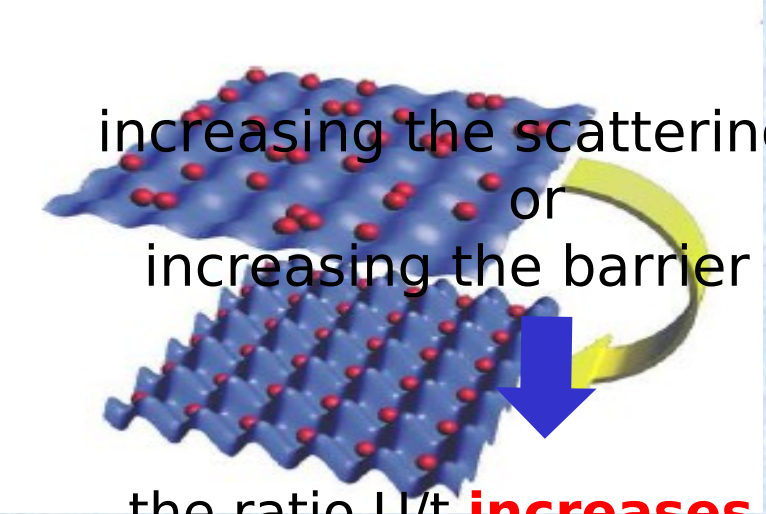
$$\hat{H} = -t \sum_{<i,j>} \left(\hat{b}_i^+ \hat{b}_j + h.c. \right) + \frac{U}{2} \sum_i \hat{n}_i \left(\hat{n}_i - 1 \right)$$

$\hat{n}_i \equiv \hat{b}_i^+ \hat{b}_i$ $\quad N_T$ *number of particles on* N *sites* $\quad$ *filling* $f = \frac{N_T}{N}$

Bose-Hubbard Hamiltonian

t/U>>1 → **Superfluid**
dynamics described by the discrete nonlinear Schroedinger equation [A. Trombettoni and A. Smerzi, PRL (2001)]

t/U<<1 → **Mott insulator** quantum fluctuations dominate



the ratio U/t **increases**

Effects of attractive interaction for a two-species Fermi gas

$$\hat{H} = -t \sum_{\sigma=\uparrow,\downarrow} \sum_{<i,j>} \left(\hat{c}^{+}_{i,\sigma} \hat{c}_{j,\sigma} + h.c. \right) - U \sum_{i} n_{i,\uparrow} n_{i,\downarrow}$$

As shown in [S. Sorella and E. Tosatti, Europhys. Lett (1992); E. Zhao and A. Paramekanti, PRL (2006) and I. Affleck and].B. Marston, PRB (1988)]**, at half filling for the honeycomb there is superfluid behaviour only at a finite value of U: this is due to the vanishing density of states at the Dirac point.**

At this critical value, a semimetal-superfluid transition takes place (i.e., a gap opens).

Mean-field results for the 3D case

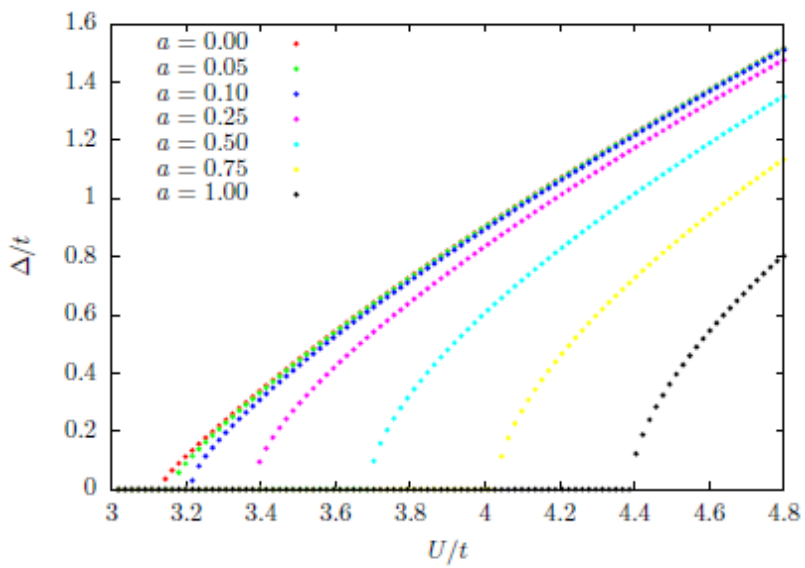


Figure 12. Gap Δ vs. U for different values of the anisotropy parameter a for the π-flux cubic lattice model at half filling. From top to bottom is $a = 0, 0.25, 0.5, 0.75, 1$.

(here $t_x = t_y = t$ and $a = t_z/t$)

Critical exponent at U_c is 1 in 3D and ½ in 2D (and 1 for a non-vanishing).

[G. Mazzucchi, L.Lepori, and A. Trombettoni, J. Phys. B (2013)]