Topological Kondo effect from Majorana fermion devices

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Content of this course

1. Majorana fermions and Majorana bound states (MBSs): Basics
2. Kitaev chain & realization in nanowires
3. Majorana takes charge
   Coupling Cooper pair and Majorana dynamics through Coulomb charging energy
4. Topological Kondo effect
   Overscreened multi-channel Kondo physics with interacting MBSs
5. Recent developments

Further reading

- J. Alicea, *Rep. Prog. Phys.* **75**, 076501 (2012)

  [many pictures used in my course are from here]
- M. Leijnse & K. Flensberg, *Semicond. Sci. Technol.* **27**, 124003 (2012)
- C.W.J. Beenakker, *Annu. Rev. Condens. Matter Phys.* **4**, 113 (2013)
- A. Zazunov, A. Altland & R. Egger, *New J. Phys.* **16**, 015010 (2014)

Part I: Introduction to Majorana fermions and MBSs

- What are Majorana fermions and Majorana bound states (MBSs)?
- How are they described?
- How can they be realized?
- What properties do they have?
- Why should we care?

What are Majorana fermions ?

- Majorana fermion is its own antiparticle $\gamma = \gamma^{+}$
  - carries no charge
  - real-valued solution of relativistic Dirac equation
- Elementary particle?

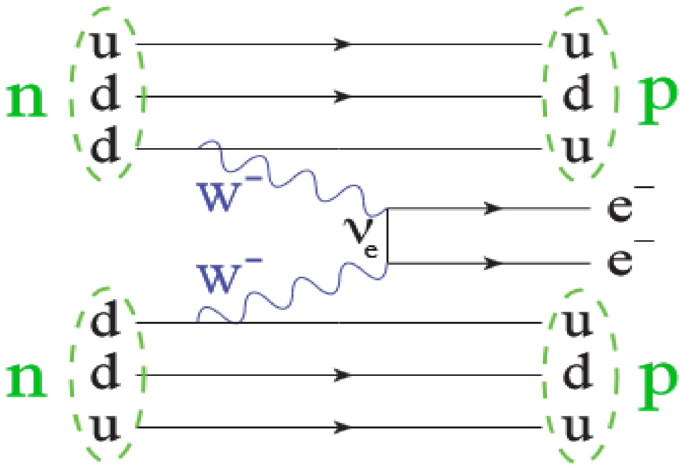
  Perhaps neutrino ?

  Double beta decay:

  For neutrino = antineutrino,

  annihilation possible...

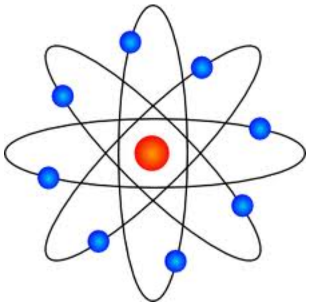
  Experiments remain unclear



- Here: search for Majorana fermions as emergent condensed matter quasiparticles

Usual (Dirac) fermions…

- Pauli principle: each single-particle state can be only filled by zero or one electron
  - Eigenstates: $|0\rangle, |1\rangle$



- Fermion operator in 2nd quantization

$$c^+|0\rangle = |1\rangle, \quad c|0\rangle = 0 \qquad cc^+ + c^+c = 1$$

$$c^+|1\rangle = 0, \quad c|1\rangle = |0\rangle \qquad c^2 = 0$$

Operator c annihilates particle (creates antiparticle)

Occupation number operator: $n = c^+c$

$n^2 = n$ $\Longrightarrow$ only eigenvalues 0,1

Majorana bound state (MBS)

- 1st quantization: $H\Psi = E\Psi$
- 2nd quantization: $[H, c^\dagger] = Ec^\dagger, \quad [H, c] = -Ec$

What about Majorana fermions? $\gamma = \gamma^+$

$[H, \gamma] = E\gamma = -E\gamma \rightarrow$ E=0 (relative to chemical potential)

- MBS = equal-weight superposition of electron and hole states, zero mode (E=0)

(unlike exciton = bosonic e-h product state)

→ search in superconductors (SCs)

NB: For bosons, particle = antiparticle is standard situation (photons!)

For fermions, nontrivial statement !

Counting Majorana state occupations

Consider set of MBSs at different locations in space

- Self-adjoint operators $\gamma_j = \gamma_j^+$
- Clifford algebra $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}$
- Different Majorana operators anticommute just like fermions
- But: $\gamma_j^+ \gamma_j = \gamma_j^2 = 1$
  - annihilation of particle & antiparticle recovers previous state
  - Occupation number of single MBS is ill-defined

So there is no Majorana sea (unlike Fermi sea) ... or perhaps there is?

Counting Majorana fermions

Count state of spatially separated MBS pair:

Non-local auxiliary fermion $c = (\gamma_1 + i\gamma_2)/2$

$$n = c^+ c = (i\gamma_1\gamma_2 + 1)/2 = 0,1 \qquad i\gamma_1\gamma_2 = 2c^+ c - 1$$

$$\gamma_1 = c + c^+$$

$$\gamma_2 = -i(c - c^+)$$

MBS = „half a fermion“,
fractionalized zero mode

U(1) gauge freedom implies equally possible choice:

$$c = e^{-i\vartheta}(\gamma_1 + i\gamma_2)/2$$

Entanglement ? **[see talk by S. Plugge]**

Semenoff & Sodano, Electron. J. Phys. (2006)
Plugge, Zazunov, Sodano & Egger, PRB (2015)

MBS in p-wave superconductors

- Bogoliubov quasiparticles in s-wave BCS SC

$$\gamma = uc_{\uparrow}^{+} + vc_{\downarrow} \neq \gamma^{+}$$

  - At Fermi level: u=v
  - Far away from Fermi level:

    either u→1 & v→0 or v→1 & u→0

    [purely electron- or hole-like]

  - But spin spoils it: no MBS possible for s-wave SC!
- better: spinless quasiparticles in p-wave SC
  - at Fermi level: $\gamma = uc^{+} + u^{*}c = \gamma^{+}$
  - Vortex in 2D p-wave SC hosts MBS
  - Experimentally most promising route (at present):

  MBS end states of 1D p-wave SC (Kitaev chain)

Kitaev chain:
„toy model“ for 1D p-wave SC

Tight-binding chain of spinless fermions

$$H = -\frac{1}{2}\sum_{j=1}^{N-1}\left(tc_j^+ c_{j+1} + \Delta e^{i\phi} c_j c_{j+1} + h.c.\right) - \mu\sum_{j=1}^{N} c_j^+ c_j$$

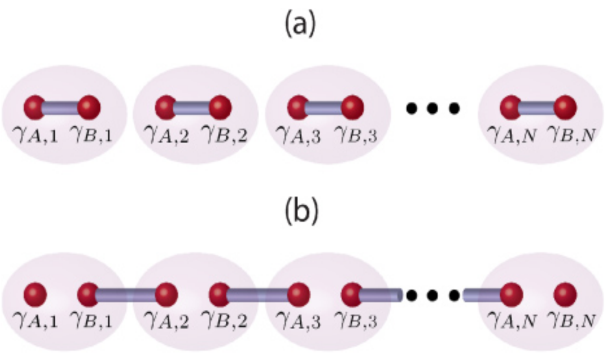
- Proximity-induced pairing gap Δ
  - In 1D only fluctuating intrinsic SC → induce pairing by proximity to bulk SC
- Hopping amplitude t>0, chemical potential μ

Majorana representation

Consider N lattice sites, open boundary conditions

- To simplify algebra, first put Δ=t and μ=0
- Decompose lattice fermions into Majorana fermions $c_j = e^{-i\phi/2}\left(\gamma_{B,j} + i\gamma_{A,j}\right)/2$
- short calculation gives

$$H = -i\frac{t}{2}\sum_{j=1}^{N-1}\gamma_{B,j}\gamma_{A,j+1}$$



- MBSs at the ends don't appear!

$$\gamma_L \equiv \gamma_{A,1}$$

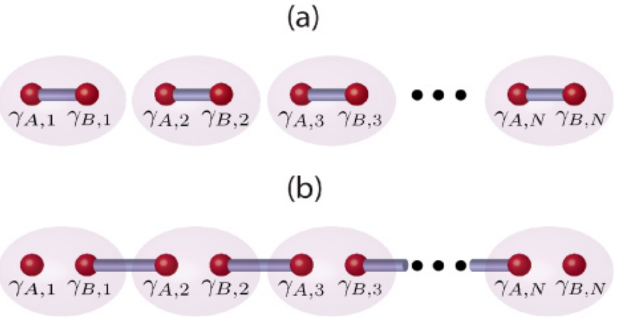
$$\gamma_R \equiv \gamma_{B,N}$$

zero modes $\left[\gamma_L, H\right] = \left[\gamma_R, H\right] = 0$

Kitaev chain: Majorana end states

- Switch to new d fermions „shifting register“

$$d_j = \left(\gamma_{B,j} - i\gamma_{A,j+1}\right)/2$$



- H diagonalized

$$H = t\sum_{j=1}^{N-1}\left(d_j^+ d_j - 1/2\right)$$

$$-i\gamma_{B,j}\gamma_{A,j+1} = 2d_j^+ d_j - 1 = \pm 1$$

- Nonlocal fermionic zero mode

$$f = \left(\gamma_L + i\gamma_R\right)/2$$

represents decoupled MBSs at ends, zero energy

Topological degeneracy

- All d-fermion states unoccupied in ground state (GS)
- Zero mode causes twofold GS degeneracy

$$|GS\rangle_E = |0\rangle_f \prod_{j=1}^{N-1} |0\rangle_j$$

$$|GS\rangle_O = |1\rangle_f \prod_{j=1}^{N-1} |0\rangle_j = f^+ |GS\rangle_E = \gamma_L |GS\rangle_E$$

Both GSs differ in fermion parity (even/odd)

Topological degeneracy

Expectation values of local operators

- Arbitrary local operator A has locally indistinguishable expectation values
  (up to exponentially small corrections)

$$ {}_E\langle GS | A | GS \rangle_E = {}_O\langle GS | A | GS \rangle_O $$

- Proof:
  - Local operator has finite support $A \sim c_i^+ c_j^+ \cdots c_k c_l \cdots$
  - Rewrite A in terms of d fermions (and possibly f)

$$ A \sim d_{i'}^+ d_{j'}^+ \cdots d_{k'} d_{l'} \cdots (f, f^+) $$

    - f appears iff A has support near a boundary

Nonlocal operators

- If A has no support near boundary:

same expectation values since $|GS\rangle_O = \gamma_L |GS\rangle_E$

$$\gamma_L^2 = 1$$

- Otherwise A has only support, say, near left boundary $A \sim \gamma_L$

Use again $\gamma_L^2 = 1$ $\rightarrow$ same expectation values for both GSs

$\rightarrow$ only nonlocal operators can distinguish [or change $\rightarrow$ topological protection] the GSs

$\rightarrow$ Basis for topological quantum computation

Kitaev chain: Arbitrary parameters

- Topological phase persists for finite (not too large) μ and/or arbitrary Δ/t (see later)
  - MBS wavefunction: Exponential decay into bulk on lengthscale ξ
  - Chain length *L* determines overlap between left/right MBS wavefunctions

    → MBS hybridization $\varepsilon_f \sim e^{-L/\xi}$

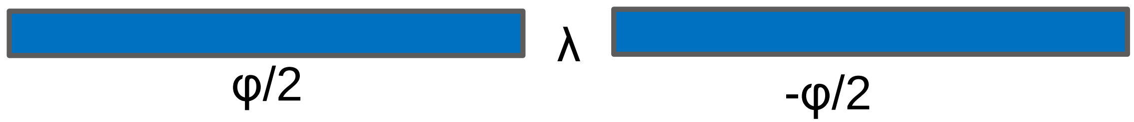
  Then: exponentially small but finite-energy mode instead of true zero mode

$$H = i\varepsilon_f \gamma_L \gamma_R = \varepsilon_f \left(2f^+ f - 1\right) \rightarrow \pm \varepsilon_f$$

Fractional Josephson effect

- Topological degeneracy crucial ingredient for hallmark experiment of MBS physics: fractional Josephson effect
- First: brief reminder of standard Josephson effect in conventional s-wave BCS superconductors

Reminder: Josephson effect



- Tunnel contact (tunneling amplitude λ) separates s-wave SCs, phase difference φ
- Tunneling of Cooper pairs (2e) gives 2π periodic Josephson energy

$$E_{Jos}(\varphi) = -E_J \cos\varphi \quad \text{with} \quad E_J \sim \lambda^2$$

- Josephson DC supercurrent-phase relation

$$I(\varphi) = \frac{2e}{\hbar}\frac{dE_{Jos}}{d\varphi} = I_c \sin\varphi \;\; \text{with} \quad I_c \sim \lambda^2$$

Now topological case

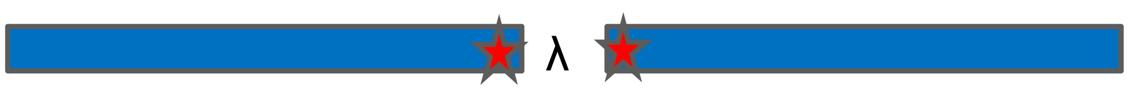
- Two tunnel-coupled Kitaev chains (Δ=t, μ=0)
  - Boundary fermions connected by tunneling

$$H_{tun} = \lambda c_L^+ c_R + h.c.$$

  - Insert effective low-energy form

$$c_L = e^{-i\phi_L/2}\left(\gamma_{B,L} + i\gamma_{A,L}\right)/2 \rightarrow e^{-i\varphi/4}\gamma_L/2$$

$$c_R = e^{-i\phi_R/2}\left(\gamma_{B,R} + i\gamma_{A,R}\right)/2 \rightarrow ie^{+i\varphi/4}\gamma_R/2$$



$\gamma_L \equiv \gamma_{B,L}$ $\quad$ $\gamma_R \equiv \gamma_{A,R}$

Projection to low-energy space

- Low energy space is spanned by MBSs →

$$H_{tun} = \frac{\lambda}{2}\cos\left(\frac{\varphi}{2}\right) i\gamma_L\gamma_R \qquad i\gamma_L\gamma_R = \pm 1$$

  - Andreev bound states (inside gap!)

$$E_{\pm}(\varphi) = \pm\frac{\lambda}{2}\cos\left(\frac{\varphi}{2}\right)$$

  - Fractional Josephson effect:

$$I(\varphi) = \frac{2e}{\hbar}\frac{dE_{\pm}}{d\varphi} = \pm\frac{e\lambda}{2\hbar}\sin\left(\frac{\varphi}{2}\right)$$

  - tunneling of „half a Cooper pair"
    → 4π periodic Josephson current-phase relation

Fractional Josephson effect

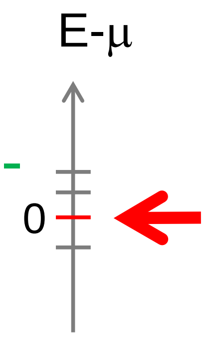
- Josephson effect via single-electron tunneling through zero mode
  - Highly unusual: supercurrent proportional to λ
- Two branches for different GS parity
  - Hamiltonian has 2π periodicity
  - GS recovered only by advancing phase by 4π
  - Parity conservation crucial for 4π periodicity
  - Quasiparticle poisoning:
    boson-mediated transitions from Andreev-MBS sector to above-gap quasiparticles $\rightarrow$ flip parity

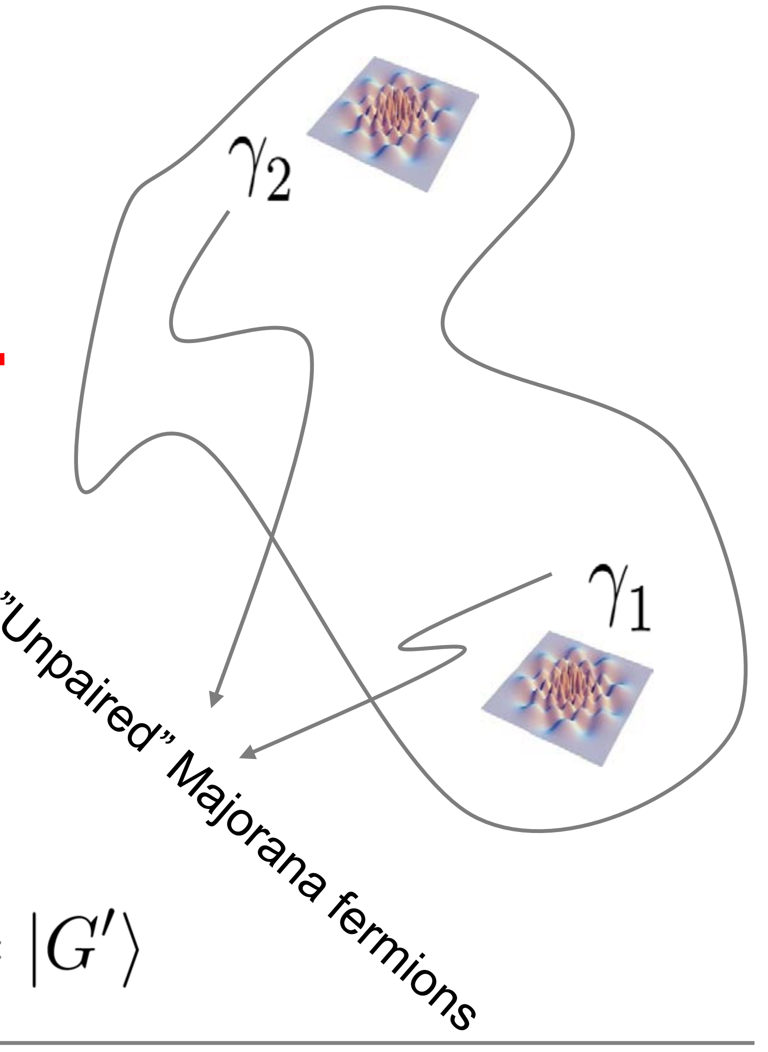
  2π periodicity restored at finite T (in stationary case)

Nonlocality and degeneracy

Spatially separate Majorana pair yields E=0 fermion mode

- Information stored non-locally & topologically protected





Ground state $|G\rangle$ is degenerate

- Even/odd number of electrons (fermion parity): same E=0
- Rotation in ground-state manifold:

$$\gamma_i|G\rangle = |G'\rangle$$

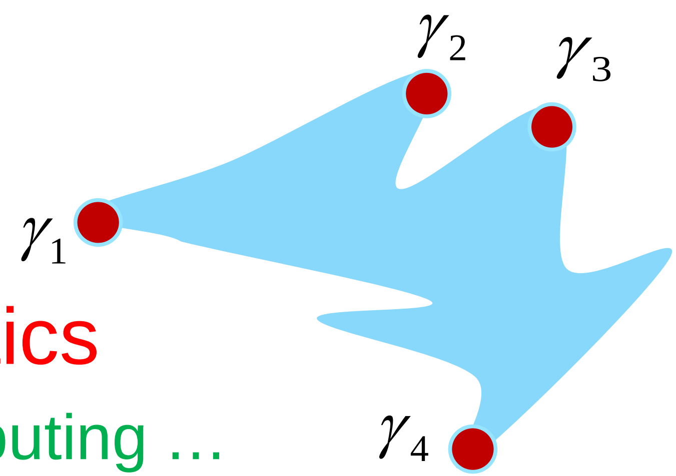
Nonabelian anyons [see lectures by Ady Stern]

Example: four MBS = two parity qubits

- Start with initial state $|G\rangle = |0_{12}, 0_{34}\rangle$
- Braiding: rotation in ground-state manifold by interchanging γ_2 and γ_3

$$U_{23}|G\rangle = \frac{1}{\sqrt{2}}\left(1+\gamma_3\gamma_2\right)|G\rangle$$

Ivanov, PRL 2001



entangled state,

nonabelian exchange statistics

…could be useful for quantum computing …

Summary of Part I

- Basic features of Majorana „fermions“
  - Fractionalized zero mode „particles“
  - Counting MBS pairs via nonlocal fermions
  - Topological degeneracy, ground-state parity
- Realizable as end states of 1D p-wave SC: Kitaev chain
- Signatures: fractional Josephson effect, nonabelian exchange statistics, ...

Part II: Kitaev chain

1. Bulk 1D p-wave superconductor (SC)

$$H=-\frac{1}{2}\sum_{j=1}^{N-1}\left(tc_j^+c_{j+1}+\Delta e^{i\phi}c_jc_{j+1}+h.c.\right)-\mu\sum_{j=1}^{N}c_j^+c_j$$

Majorana end states reflect bulk topology:

bulk-boundary correspondence

- Sensitivity of ground state to boundary conditions
- Bulk topological index

2. Kitaev chain can be realized in lab

Semiconductor nanowires with strong spin-orbit coupling, Zeeman field, proximity coupled to conventional s-wave SC

Bulk topology

MBSs mirror bulk topological features →
consider ring: periodic BCs (arbitrary parameters)

$$H = \frac{1}{2}\sum_{k\in BZ} C_k^+ \underbrace{\begin{pmatrix} \xi_k & \Delta_k^* \\ \Delta_k & -\xi_k \end{pmatrix}}_{H_{BdG}} C_k$$

[1/2 : no double counting!]

$$c_j = \frac{1}{\sqrt{N}}\sum_{k\in BZ} e^{ikj}\tilde{c}_k$$

$$C_k = \begin{pmatrix} \tilde{c}_k \\ \tilde{c}_{-k}^+ \end{pmatrix}$$ **Nambu spinor**

$$\xi_k = -t\cos k - \mu$$

kinetic energy

$$\Delta_k = -i\Delta e^{i\phi}\sin k = -\Delta_{-k}$$

Fourier transformed p-wave pairing potential

BdG equation

Diagonalize Hamiltonian $$H = \sum_{k \in BZ} E_k a_k^+ a_k$$

Quasiparticle operators $$a_k = u_k \tilde{c}_k + v_k \tilde{c}_{-k}^+$$

Bogoliubov-deGennes (BdG) equation

$$\begin{pmatrix} \xi_k - E_k & \Delta_k^* \\ \Delta_k & -\xi_k - E_k \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = 0$$

solved by $$u_k = \frac{\Delta_k}{|\Delta_k|} \sqrt{\frac{E_k + \xi_k}{2E_k}}$$

$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2} \qquad v_k = \frac{E_k - \xi_k}{\Delta_k} u_k$$

Phase diagram of Kitaev chain

Topological phase transitions require gap closing $E_k = 0 \;\rightarrow\; \Delta_k = \xi_k = 0$

two solutions: $k = 0 \;\text{ with }\; \mu = -t$

$k = \pm\pi \;\text{ with }\; \mu = t$

- $|\mu| > t$: topologically trivial „strong pairing“ phase, adiabatically connected to vacuum

  $\mu < -t$ and $\mu > t$ phases related by e-h symmetry

- Topologically nontrivial „weak pairing“ regime (with MBSs under open BCs) contains μ=0 → corresponds to $|\mu| < t$

Topological superconductor

- BdG Hamiltonian: $H_{BdG} = \vec{B}(k) \cdot \vec{\tau}$
  - Nambu „spin“ in „magnetic field“
  - particle-hole symmetry requires: $B_{x,y}(k) = -B_{x,y}(-k)$

    $B_z(k) = B_z(-k)$

$\longrightarrow$ field needed only for $0 \le k \le \pi$

- Within a gapped phase: study map from BZ to unit sphere $k \to \hat{B} = \vec{B}(k) / |\vec{B}(k)|$

values at k=0 and k=π restricted by

note that $\Delta_{k=0,\pi} = 0$

$$\hat{B}(0) = s_0 \hat{z}$$

$$s_{0,\pi} = \mathrm{sgn}(\xi_{k=0,\pi}) = \pm 1$$

$$\hat{B}(\pi) = s_\pi \hat{z}$$

Z_2 topological invariant

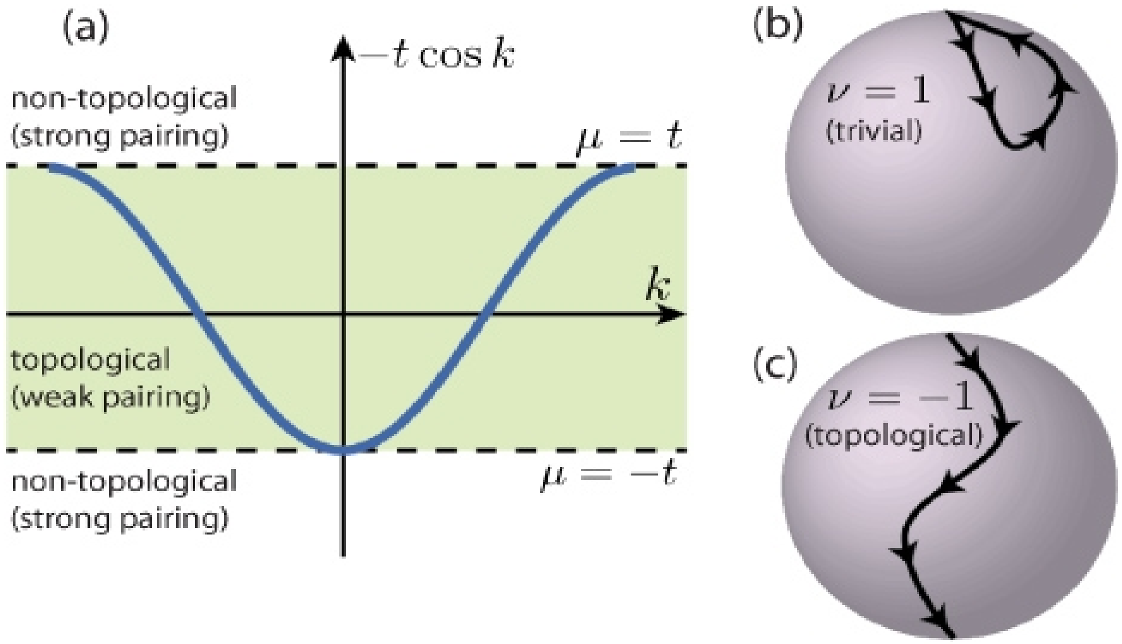
Follow field direction from k=0 to k=π

Either field stays near same pole (top. trivial)

or explores whole sphere (top. nontrivial)

Z_2 invariant

$$\nu = s_0 s_\pi = \mathrm{sgn}\left(\xi_0 \xi_\pi\right)$$



Ground state: elementary derivation

(for μ≈-t)

$$H = \xi_{k=0}\tilde{c}_{k=0}^{+}\tilde{c}_{k=0} + \sum_{k>0}\left(\xi_k\left(\tilde{c}_k^{+}\tilde{c}_k + \tilde{c}_{-k}^{+}\tilde{c}_{-k}\right) + \left[\Delta_k^{*}\tilde{c}_k^{+}\tilde{c}_{-k}^{+} + h.c.\right]\right)$$

note $\Delta_{k=0} = 0$

(k,-k) pairs decouple

.. use basis states $\left|n_k, n_{-k}\right\rangle$

Solve for each k (decoupled even/odd parity sector)

$$H_{k,even} = \begin{pmatrix} 0 & \Delta_k \\ \Delta_k^{*} & 2\xi_k \end{pmatrix}$$ basis $\left|00\right\rangle, \left|11\right\rangle$ $$E_{k,\pm}^{even} = \xi_k \pm \sqrt{\xi_k^2 + \left|\Delta_k\right|^2}$$

$$H_{k,odd} = \begin{pmatrix} \xi_k & 0 \\ 0 & \xi_k \end{pmatrix}$$ basis $\left|01\right\rangle, \left|10\right\rangle$ $$E_k^{odd} = \xi_k$$

Lowest energy has even parity $\rightarrow$ $$\left|GS\right\rangle \sim \prod_{k>0}\left(u_k\left|00\right\rangle + v_k\left|11\right\rangle\right)$$

Sensitivity to boundary conditions

k=0 unpaired fermion mode at $\xi_{k=0} = -t - \mu$

- μ > -t: Mode occupied $\rightarrow$ odd parity GS
- Antiperiodic boundary conditions:
  no k=0 mode exists, even parity GS

Sensitivity to boundary conditions indicates topologically nontrivial phase

No such sensitivity for μ < -t

- Then always even parity GS:
  topologically trivial phase

Interpolate between boundary conditions

Consider t→λ for one link of a Kitaev ring in topological phase:

- λ= - t: antiperiodic BC
- λ = 0: open BC
- λ=+ t: periodic BC

- Changing λ from –t to +t, one must go through degenerate GS (with opposite fermion parity)
  - otherwise GS nondegenerate with finite gap

Long-wavelength continuum limit

- BdG Hamiltonian for small k:

$$H_{BdG} = \begin{pmatrix} -t-\mu & 2ik\Delta \\ 2ik\Delta & t+\mu \end{pmatrix}$$

NB. dropping k^2 terms is controlled approximation

- Construction of MBS: Consider spatially varying chemical potential $\mu(x) = -t + \alpha x$

$$H_{BdG} = \begin{pmatrix} -\alpha x & 2\Delta\partial_x \\ -2\Delta\partial_x & \alpha x \end{pmatrix} = -\alpha x \tau_z - 2p\Delta\tau_y$$

$$p = -i\partial_x$$

Squaring trick

- To obtain spectrum, square BdG Hamiltonian

$$H_{BdG}^2 = \alpha^2 x^2 + 4\Delta^2 p^2 + 2\alpha\Delta(xp - px)(-i\tau_x)$$
$$= \alpha^2 x^2 + 4\Delta^2 p^2 + 2\alpha\Delta\tau_x$$

  - Choose Nambu basis: $\tau_x \to \pm 1$
  - 1D harmonic oscillator: $\frac{1}{2m} \to 4\Delta^2, \ \frac{1}{2} m\omega^2 \to \alpha^2$
  - Frequency: $\omega = 4\Delta\alpha$
- Eigenenergies (n=0,1,2,...)

$$E_{n,\pm}^2 = \hbar\omega\left(n + 1/2 \pm 1/2\right)$$

Majorana bound state

- Zero energy solution $E_{0,-} = 0$ $\phi_0(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$
- Localized around transition point x=0:

$$u, v \sim e^{-x^2/2l^2}, \; l = \sqrt{\hbar/m\omega} = \sqrt{2\hbar\Delta/\alpha}$$

- BdG states: particle-hole symmetry encoded in $\left[\tau_x, H_{BdG}\right]_+ = 0$. This implies $\phi_E \to \phi_{-E} = \tau_x \phi_E$
  - Majorana state at E=0 has $u = v$

Explicit construction of MBS operator: $c_j \to \Psi(x)$

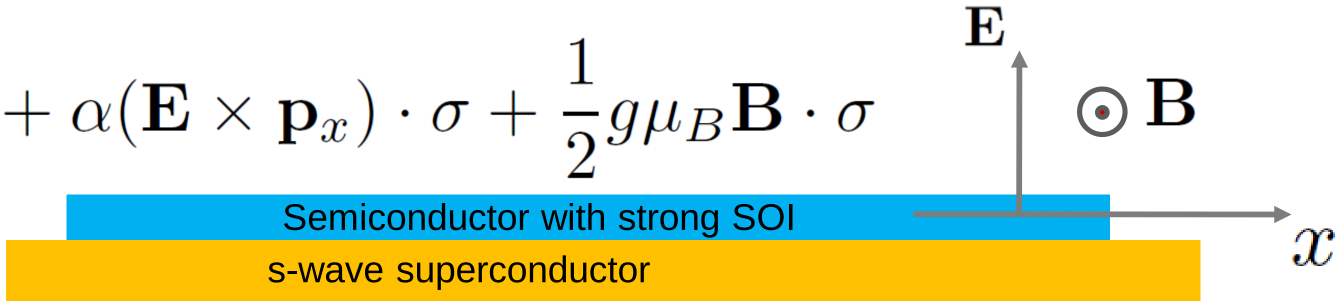
$$\gamma = \int dx \left[u(x)\Psi(x) + u(x)\Psi^+(x) \right] = \gamma^+$$

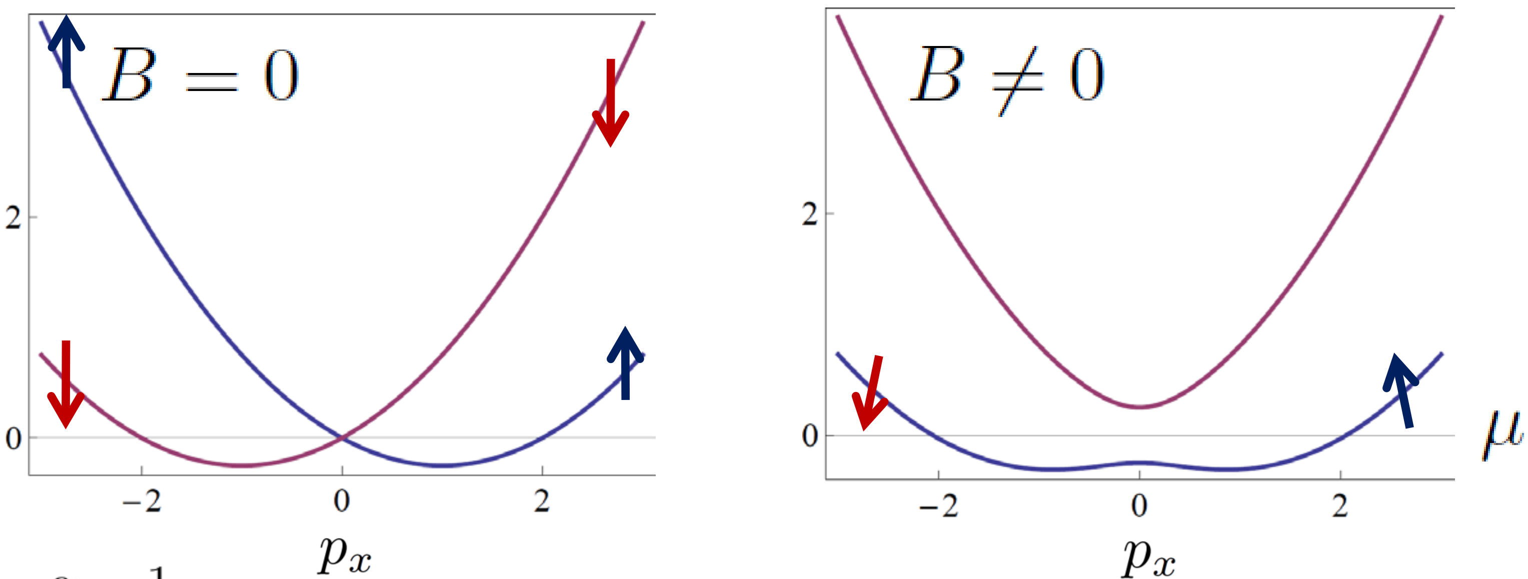
How to realize Kitaev chain in the lab?

- 1D spinless fermions: use half-metal or large Zeeman splitting?
  - but proximity effect from s-wave SCs then difficult
- Better: admixture of effective s- and p-wave pairing in 1D nanowires with
  - Strong (Rashba) spin-orbit coupling: InAs, InSb
  - Magnetic Zeeman field
    - exploit large Landé factor for InAs, InSb
    - Orientation not crucial (but not along spin-orbit axis)
  - Proximity effect from close-by conventional s-wave SC: Nb, NbTiN, ...

Rashba quantum wire (InAs, InSb)

$$H_0 = \frac{p_x^2}{2m} + \alpha(\mathbf{E} \times \mathbf{p}_x) \cdot \sigma + \frac{1}{2} g \mu_B \mathbf{B} \cdot \sigma$$



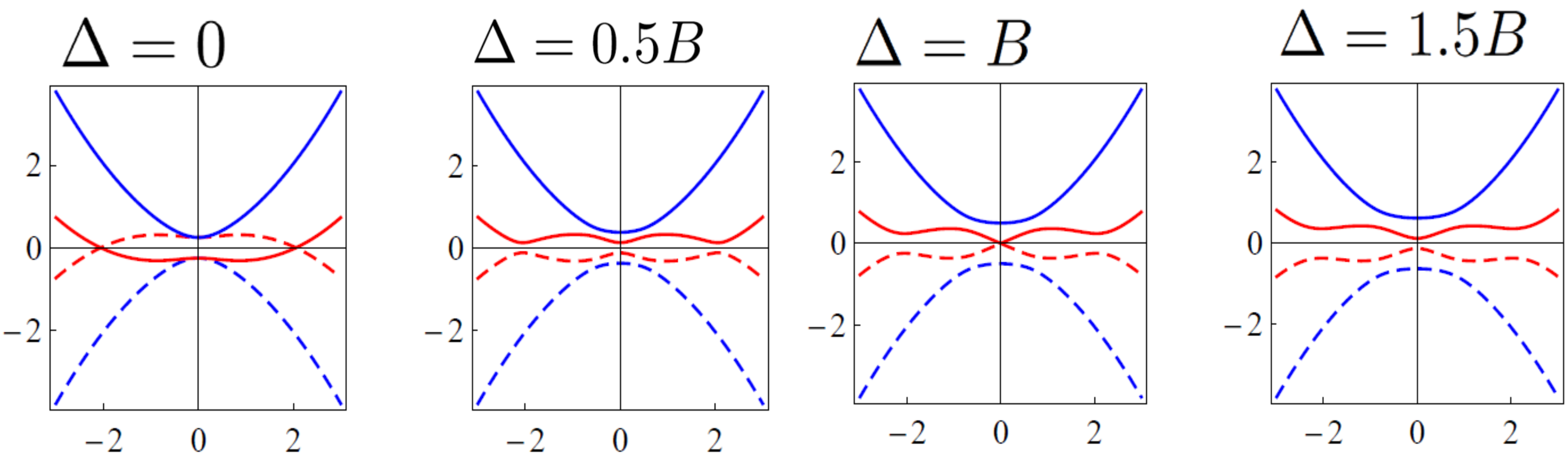


$m = \alpha = 1$

Oreg, Refael & von Oppen, PRL 2010
Lutchyn, Sau & Das Sarma, PRL 2010

1D helical liquid and proximity effect

- Without proximity coupling: 1D helical liquid
  - Spin of fermion is enslaved by momentum direction
  - Opposite momenta have (approximately) opposite spin
- Now: include coupling to s-wave superconductor
  - Gap closes and reopens at p=0: B>Δ topological phase



BdG Hamiltonian

$$H = \int dx \Psi^{+}(x) H_{BdG} \Psi(x)$$

$$\Psi = \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \\ \psi_{\downarrow}^{+} \\ -\psi_{\uparrow}^{+} \end{pmatrix}$$

- Four-spinor combines spin and Nambu space
  - Necessary because of spin-orbit coupling
  - Caution: avoid double counting!
  - „-“ sign highlights time-reversal symmetry

$$\mathrm{T} = -i\sigma_y C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C$$

$$H_{BdG} = \left[\left(\frac{p^2}{2m} - \mu \right) + up\sigma_y \right] \tau_z - B\sigma_z + \Delta \tau_x$$

Rashba field, not aligned with Zeeman field

Zeeman field

Proximity induced pairing

Dispersion

- B=Δ=0: Shifted parabolas $E_p = \xi_p \pm up$
- Δ=0: gap opens near p=0 $E_p = \xi_p \pm \sqrt{u^2 p^2 + B^2}$
  - Pair of (almost) helical states for μ in „gap“ at p=0
  - Now: μ=0 and strong spin-orbit $B << mu^2$
- Gap closing and reopening near p=0 described by $H_{BdG} = up\sigma_y\tau_z - B\sigma_z + \Delta\tau_x$
  - Squaring trick

$$H_{BdG}^2 = u^2 p^2 + B^2 + \Delta^2 - 2B\Delta\underbrace{\sigma_z\tau_x}_{=\pm 1}$$

Dispersion near p=0

- Gap closing at B=Δ signals topological phase transition

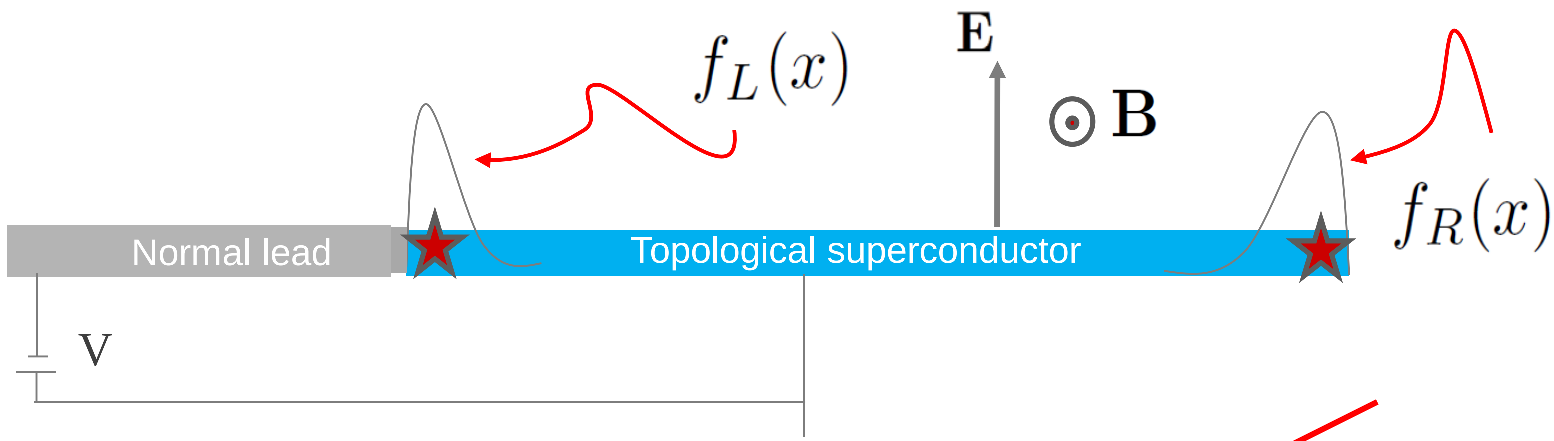
$$E_{p,\pm}^2 = u^2 p^2 + \left(B \pm \Delta\right)^2$$

- B>Δ corresponds to topological phase of Kitaev chain: Majorana end states
- For finite μ: $B_c = \sqrt{\Delta^2 + \mu^2}$
  - One can tune Zeeman field or chemical potential to reach topological regime !

How to detect Majorana states?

1. Fractional Josephson effect (but requires study of dynamics...)
2. Zero bias anomaly in tunneling conductance (or related features)
3. Nonlocal effects in interacting devices, e.g. topological Kondo physics

Zero bias anomaly (ZBA)



$$\begin{pmatrix} \Psi_{S,\uparrow}(x) \\ \Psi_{S,\downarrow}(x) \\ \Psi^\dagger_{S,\uparrow}(x) \\ \Psi^\dagger_{S,\downarrow}(x) \end{pmatrix} = \gamma_1 f_L(x) \begin{pmatrix} 1 \\ i \\ 1 \\ -i \end{pmatrix} \cancel{+ \text{quasiparticles with} \quad E \geq \Delta}$$

Tunneling into Majorana state from a normal lead

$$\gamma_1 = \int dx \; f_L(x) \left(\Psi_{\uparrow_y}(x) + \Psi^\dagger_{\uparrow_y}(x) \right)$$ Spin up along y

$$\gamma_2 = \int dx \; f_R(x) \left(\Psi_{\downarrow_y}(x) + \Psi^\dagger_{\downarrow_y}(x) \right)$$ Spin down along y

ZBA conductance peak

Tunneling Hamiltonian

$$H_T = \sum_k \left(v_{1,k} c_{k\uparrow_y} - v_{1,k}^* c_{k\uparrow_y}^\dagger \right) \gamma_1$$

Transport signature of Majoranas:

Zero-bias conductance peak due to resonant Andreev reflection

Bolech & Demler, PRL 2007

Law, Lee & Ng, PRL 2009

Flensberg, PRB 2010

$$G_{T=0}(V) = \frac{2e^2}{h} \frac{1}{1+(eV/\Gamma)^2}$$

Experimental Majorana signatures

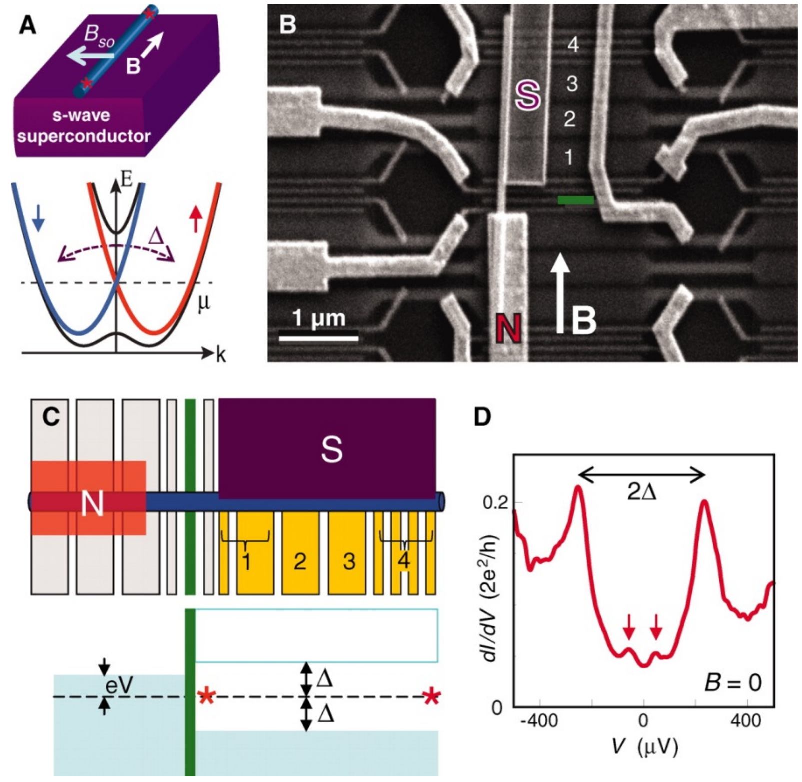
InAs or InSb nanowires expected to host Majoranas due to interplay of
• strong Rashba spin orbit field
• magnetic Zeeman field
• proximity-induced pairing

Oreg, Refael & von Oppen, PRL 2010
Lutchyn, Sau & Das Sarma, PRL 2010

Transport signature of Majoranas: Zero-bias conductance peak due to resonant Andreev reflection

Bolech & Demler, PRL 2007
Law, Lee & Ng, PRL 2009
Flensberg, PRB 2010

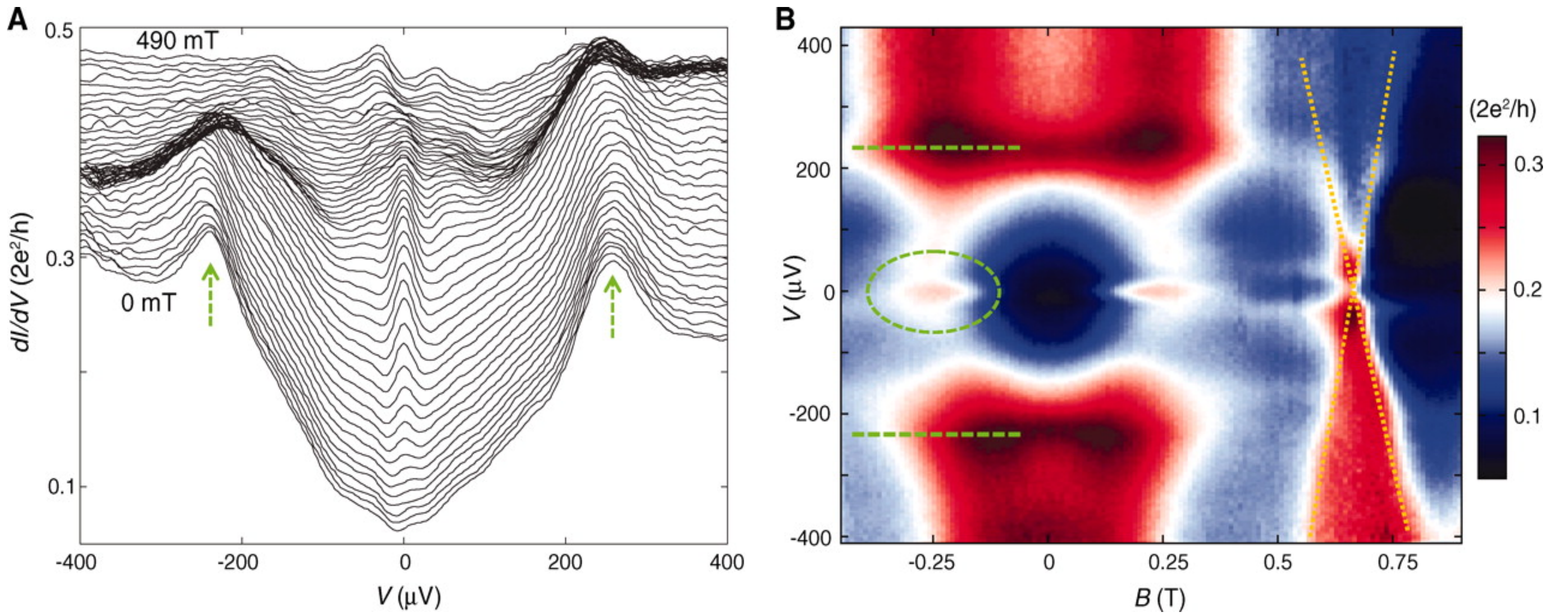
Mourik et al., Science 2012



see also: *Rokhinson et al., Nat. Phys. 2012; Deng et al., Nano Lett. 2012; Das et al., Nat. Phys. 2012; Churchill et al., PRB 2013; Nadj-Perge et al., Science 2014*

Zero-bias conductance peak

Mourik et al., Science 2012



Possible explanations:

- Majorana state (most likely)
- Disorder-induced peak — *Bagrets & Altland, PRL 2012*
- Smooth confinement — *Kells, Meidan & Brouwer, PRB 2012*
- Kondo effect — *Lee et al., PRL 2012*

Conclusions Part II

- Bulk-boundary correspondence: Kitaev chain
  - Bulk topological phase: Z_2 topological invariant, sensitivity to boundary conditions
- Realization of Kitaev chain in semiconductor nanowires with
  1. strong spin-orbit coupling
  2. sufficiently (but not too) strong magnetic Zeeman field
  3. and proximity-induced superconductivity
- Experimental signature: Zero-bias anomaly in tunneling conductance
  - resonant Andreev reflection

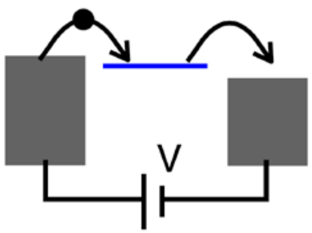
Part III: Majorana takes charge

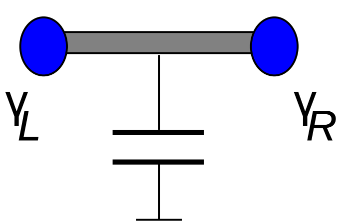
- So far (effectively) noninteracting problem
- Effect of e-e interaction on Majorana fermions
  - Interactions couple Majorana and Cooper pair dynamics
  - Consider charging energy in floating (not grounded) device hosting MBSs
  - Results in novel nonlocal effects
- Simplest case: Majorana single-charge transistor

Fu, PRL 2010; Hützen, Zazunov, Braunecker, Levy Yeyati & Egger, PRL 2012

Transport beyond ZBA

- Coulomb interactions: floating device
- Simplest: Majorana single-charge transistor
  - Overhanging helical wire parts: normal leads tunnel-coupled to MBSs
  - Nanowire part in proximity to superconductor hosts two MBSs
  - Include charging energy of floating Majorana island
  - Low energy: no quasiparticles
  - For now assume no MBS overlap





Charging energy

Two zero modes:

1. Majorana bound states

$$f = (\gamma_L + i\gamma_R)/2$$

$$2f^+ f - 1 = i\gamma_L\gamma_R = \pm 1$$

2. Cooper pair number & conjugate superconductor phase

$$[N_c, \varphi]_- = -i$$

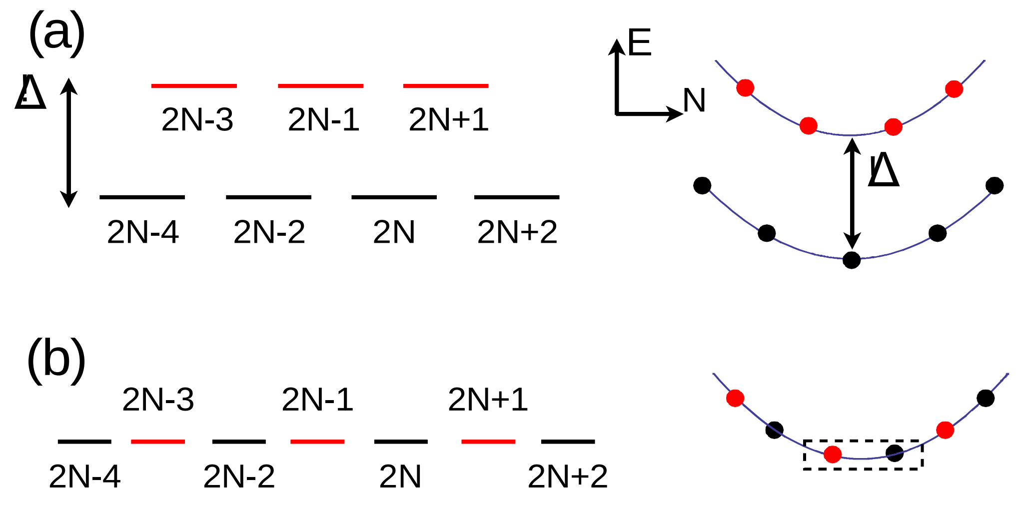
$$H_{island} = E_C\left(2N_c + f^+ f - n_g\right)^2$$

(gate parameter n_g)

Majorana fermions couple to Cooper pairs through the charging energy

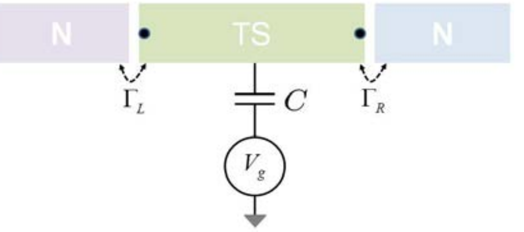
Absence of even-odd effect

- Without MBSs: Even-odd effect
- With MBSs: no even-odd effect!
  - Tuning wire parameters into the topological phase removes even-odd effect



Fu, PRL 2010

Leads & Tunneling Hamiltonian



- Normal lead tunnel-coupled to MBS
  - Can be described as spinless helical wire
    - Applied bias voltage = chemical potential difference
- Electron tunneling from lead to island
  - Low energies: tunneling only proceeds via MBS
  - Project electron operator in TS to Majorana sector
  - MBS spin structure contained in tunneling amplitude

Tunneling Hamiltonian

Source (drain) couples to left (right) MBS only.

First guess:

$$H_T = \sum_{j=L,R} t_j c_j^+ \gamma_j + h.c.$$

But: charge conserved in floating device!

- Hybridizations between leads and island: $\Gamma_j \sim |t_j|^2$
- Linewidth of zero mode: $\Gamma = \Gamma_L + \Gamma_R$

Re-express using f fermion &

take charge conservation into account:

$$H_T = t_L c_L^+ \left(f + e^{-i\varphi} f^+ \right) - i t_R c_R^+ \left(f - e^{-i\varphi} f^+ \right) + h.c.$$

Cooper pair splitting operator

Gauge choice

Using different gauge

$$f = e^{-i\varphi/2}\left(\gamma_L + i\gamma_R\right)/2$$

instead gives

$$H_T = \sum_{j=L,R} t_j c_j^+ e^{-i\varphi/2}\gamma_j + h.c.$$

Majorana mode appears charge neutral in this gauge

Majorana Meir-Wingreen formula

- Exact expression for interacting case

$$I_{j=L,R} = \frac{e\Gamma_j}{h}\int dE \; F\left(E-\mu_j\right)\operatorname{Im} G^{ret}_{\gamma_j}(E)$$

  - Lead Fermi distribution encoded in $F(E) = \tanh\left(E/2T\right)$
  - Computation of retarded Majorana Green's function required

- Differential conductance:

$$G = dI/dV$$

$$I = \left(I_L - I_R\right)/2$$

Noninteracting case: Resonant Andreev reflection

Bolech & Demler, PRL 2007
Law, Lee & Ng, PRL 2009

- E_C=0: Majorana spectral function

$$-\operatorname{Im} G^{ret}_{\gamma_j}(E) = \frac{\Gamma_j}{E^2 + \Gamma_j^2}$$

- T=0 nonlinear differential conductance:

$$G(V) = \frac{2e^2}{h} \frac{1}{1 + (eV/\Gamma)^2}$$

- Currents I_L and I_R fluctuate independently, superconductor effectively grounded
- Decoupling of currents for all cumulants (FCS) in noninteracting case: Currents flow to ground

Strong blockade: Electron teleportation

Fu, PRL 2010

- Peak conductance for half-integer n_g
- Strong charging energy then allows only two degenerate charge configurations
- Model maps to spinless resonant tunneling model
- Linear conductance (T=0): $G = e^2 / h$
  - Halving of peak conductance compared to non-interacting case
- Interpretation: Electron teleportation due to nonlocality of fermion zero mode f

Crossover from resonant Andreev reflection to electron teleportation

- Semiclassical approach to phase dynamics

Zazunov, Levy Yeyati & Egger, PRB 2011

  - Practically useful in weak Coulomb blockade regime: interaction corrections to conductance

- Full crossover from three other methods:

Hützen, Zazunov, Braunecker, Levy Yeyati & Egger, PRL 2012

  - Master equation for T>Γ: include sequential and all cotunneling processes (incl. local and crossed Andreev reflection)
  - Equation of motion approach for peak conductance
  - Zero bandwidth model for leads: exact solution

Weak Coulomb blockade regime

- Phase fluctuations are small & allow for semiclassical expansion
  - no dependence on gate parameter yet
- Results in Langevin equation for phase dynamics

$$\ddot{\varphi} + \Omega\dot{\varphi} = \xi(t)$$

  - Inverse RC time of effective circuit: $\Omega = \eta E_C$
  - Dimensionless damping strength $\eta = \frac{2}{\pi}\sum_j \frac{\Gamma_j^2}{\mu_j^2 + \Gamma_j^2}$

    (higher energy scales: damping retardation!)
  - Gaussian random force $\langle \xi(t)\xi(t') \rangle = 4E_C^2 K(t - t')$

How to obtain the current…

K has lengthy expression…

- in equilibrium satisfies fluctuation dissipation theorem

$$K_{eq}(\omega)=\frac{\omega}{2}\coth\left(\frac{\omega}{2T}\right)\eta_{eq}(\omega)$$

- Current:

$$I_j=\Gamma_j\int d\tau\ G_{\gamma_j}^{ret}(\tau)\sin\left(\mu_j\tau\right)F(\tau)e^{-J(\tau)}$$

noninteracting MBS GF

$$J(t-t')=\frac{1}{2}\left\langle\left[\overline{\varphi}(t)-\overline{\varphi}(t')\right]^2\right\rangle_{\xi}\geq 0$$

solution $\overline{\varphi}(t)$ for given noise realization

- Some algebra:

$$J(\tau)=\frac{1}{\pi\eta^2}\int_0^{\infty}d\omega\ K(\omega)\frac{1-\cos\omega\tau}{\omega^2\left(1+\omega^2/\Omega^2\right)}$$

Nonlinear conductance

$$\mu_L = -\mu_R = eV/2$$
$$\Gamma_L = \Gamma_R = \Gamma/2$$

- Symmetric system @ T=0
- Observable:

$$g(V) = \frac{I(V)}{e^2 V/h}$$

  - Noninteracting case (resonant Andreev reflection):

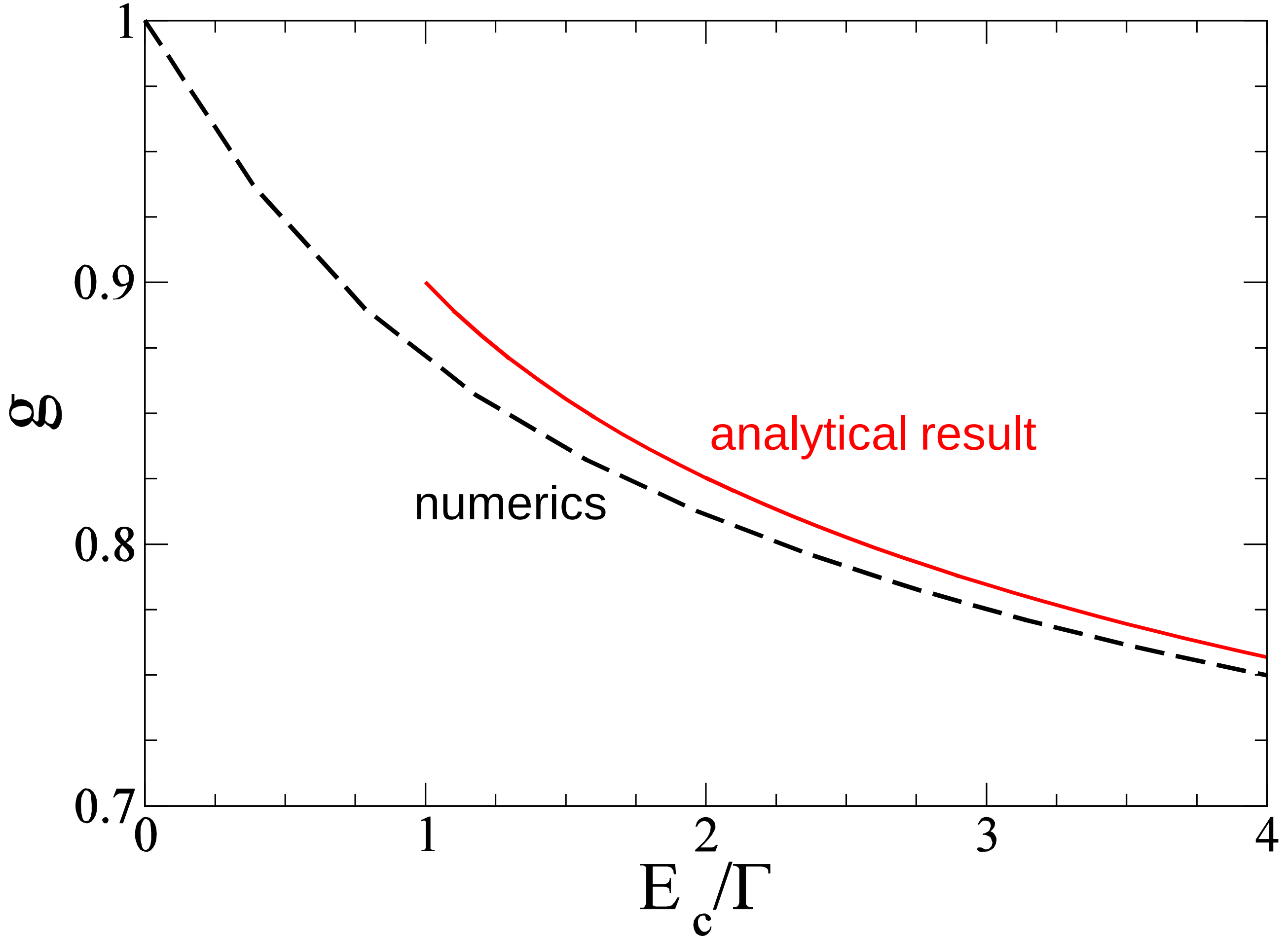
$$g^{(0)}(V) = \frac{\Gamma}{eV} \tan^{-1} \frac{eV}{\Gamma} \leq 1$$

  - Analytical result for $\Gamma < E_C$: universal power law suppression of linear conductance with increasing charging energy

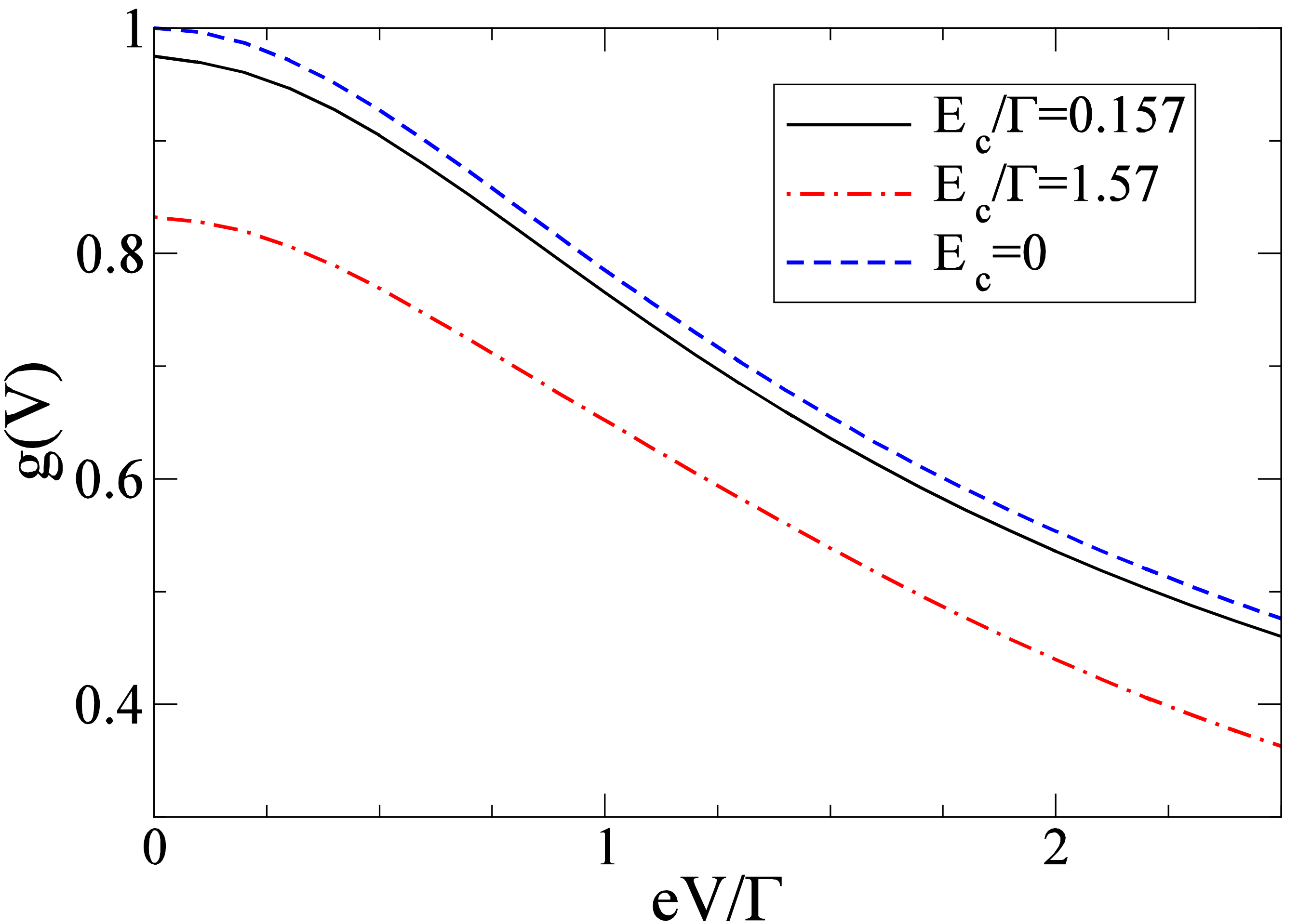
$$g(0) \approx 0.96 \left(E_C / \Gamma \right)^{-1/8}$$

Linear conductance: numerics

interaction induced suppression



Nonlinear conductance



Strong Coulomb blockade

- Strong Coulomb effects are beyond semiclassical expansion
  - Winding numbers: dependence on gate parameter
- For T, eV > Γ : master equation approach
  - Stationary probabilities for $Q = 2N_c + n_f$ particles on island obey master equation

$$\sum_{Q' \neq Q} \left[W(Q' \to Q) P(Q') - W(Q \to Q') P(Q) \right] = 0$$

  - Rates include sequential tunneling, cotunneling, and Andreev reflection processes from systematic expansion in Γ

Rates entering master equation

- Sequential tunneling processes: Golden rule

$$W(Q \to Q \pm 1) = \sum_{j=L,R} \Gamma_j \, f\left(E_{Q\pm1} - E_Q \mp \mu_j\right)$$

Fermi function

$$E_Q = E_C \left(Q - n_g\right)^2$$

- Elastic Cotunneling: transfer of electron from left to right lead by „tunneling“ through island with given Q
  - Intermediate virtual excitation of island
  - EC rates don‘t enter master equation but show up in current
  - Usually EC strongly suppressed by quasiparticle gap, but Majorana modes yield important EC contributions to conductance !

Andreev reflection (AR) rates

$$W(Q \to Q \pm 2) = W_{CAR}^{Q \to Q\pm 2} + \sum_{j=L,R} W_{j,LAR}^{Q \to Q\pm 2}$$

- **Local AR:** Electron and hole from same lead combine to form Cooper pair (or reverse process)
- **Crossed AR:** Electron and hole are from different leads
- **Example: CAR rate**

  (regularization by principal-value integration necessary)

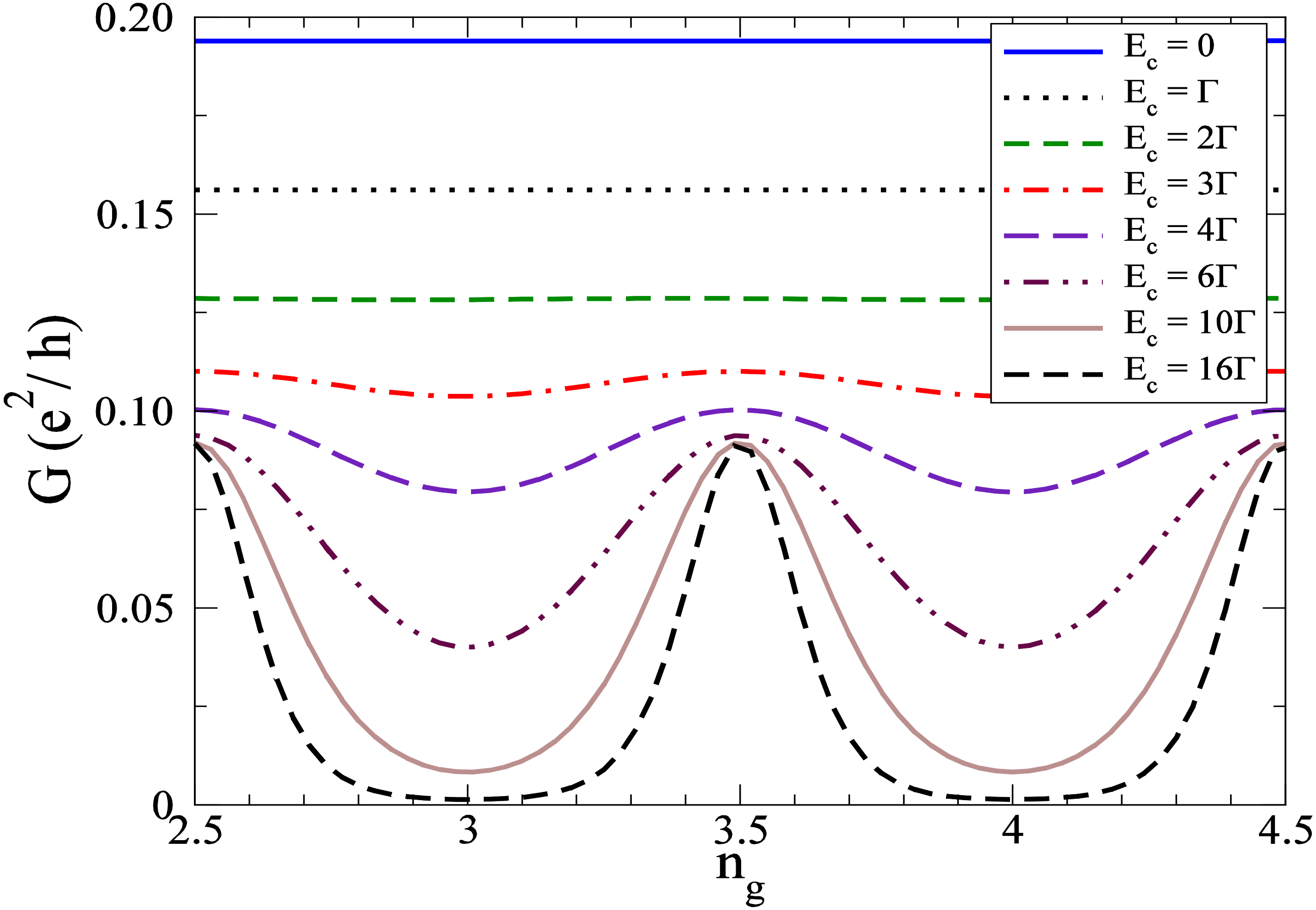
$$W_{CAR}^{Q \to Q+2} = \frac{\Gamma_L \Gamma_R}{8\pi} \int dE \int dE' f(E - \mu_L) f(E' - \mu_R) \delta\left(E + E' - \left(E_{Q+2} - E_Q\right)\right)$$

$$\times \left| \frac{1}{E - \left(E_{Q+1} - E_Q\right)} + \frac{1}{E' - \left(E_{Q+1} - E_Q\right)} \right|^2$$

Coulomb oscillations

Master equation

$T = 2\Gamma$



Valley conductance

- Analytical result for valley lineshape in strong Coulomb blockade limit $E_C >> T, \Gamma$

$$G(\delta) = \frac{e^2}{h} \frac{\Gamma_L \Gamma_R}{E_C^2} \frac{1}{\left(1 - 4\delta^2\right)^2}$$

  - Small deviation from valley center: $\delta = n_g - [n_g]$
  - Dominated by Elastic Cotunneling
  - Andreev reflection processes are strongly suppressed by Coulomb effects

Finite T conductance peak

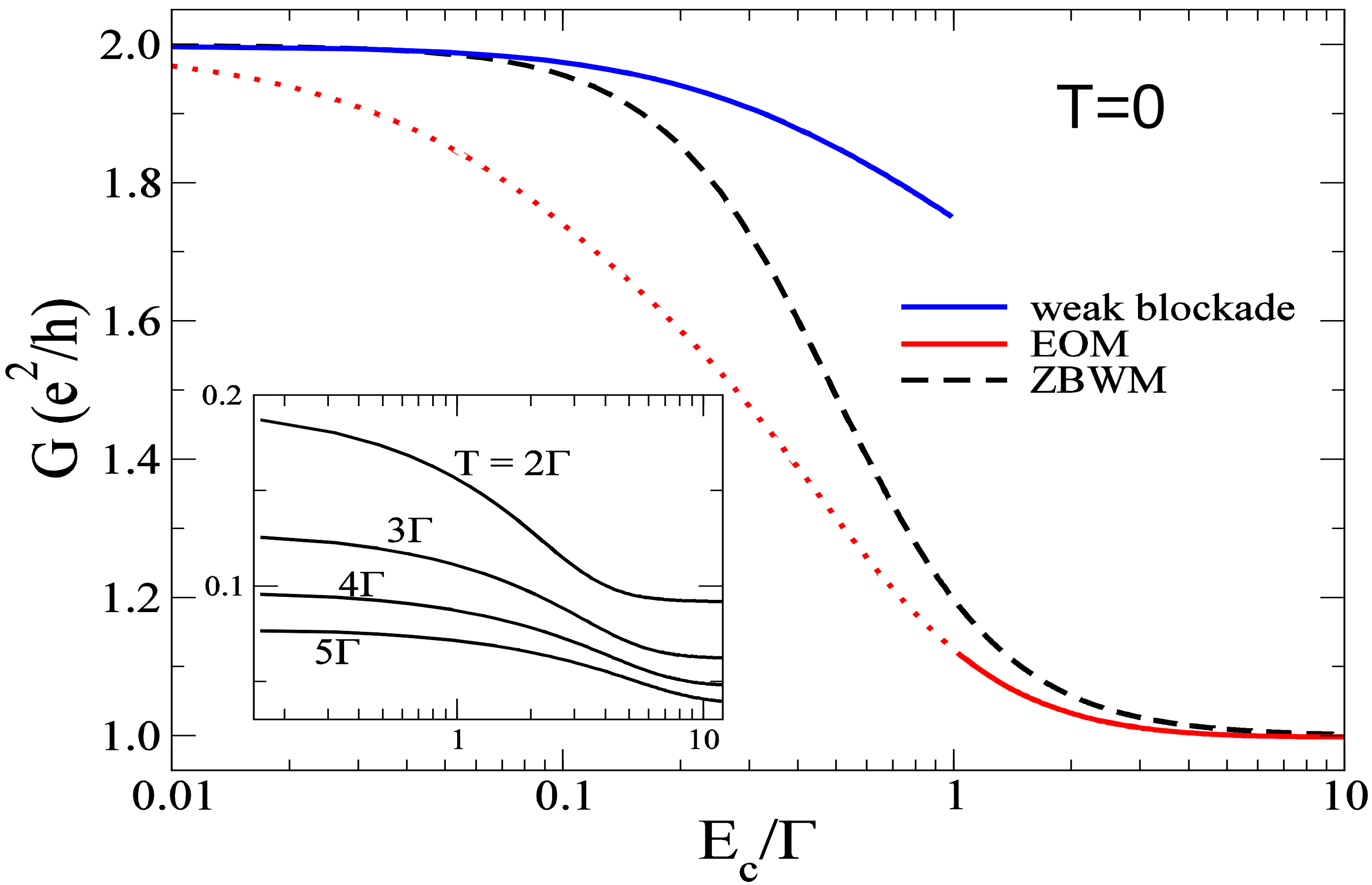
Master equation for strong charging: sequential tunneling yields peak lineshape

$$G(\delta)=\frac{e^2}{h}\frac{\pi\Gamma}{16T}\frac{1}{\cosh^2\left(\delta E_C/T\right)}$$

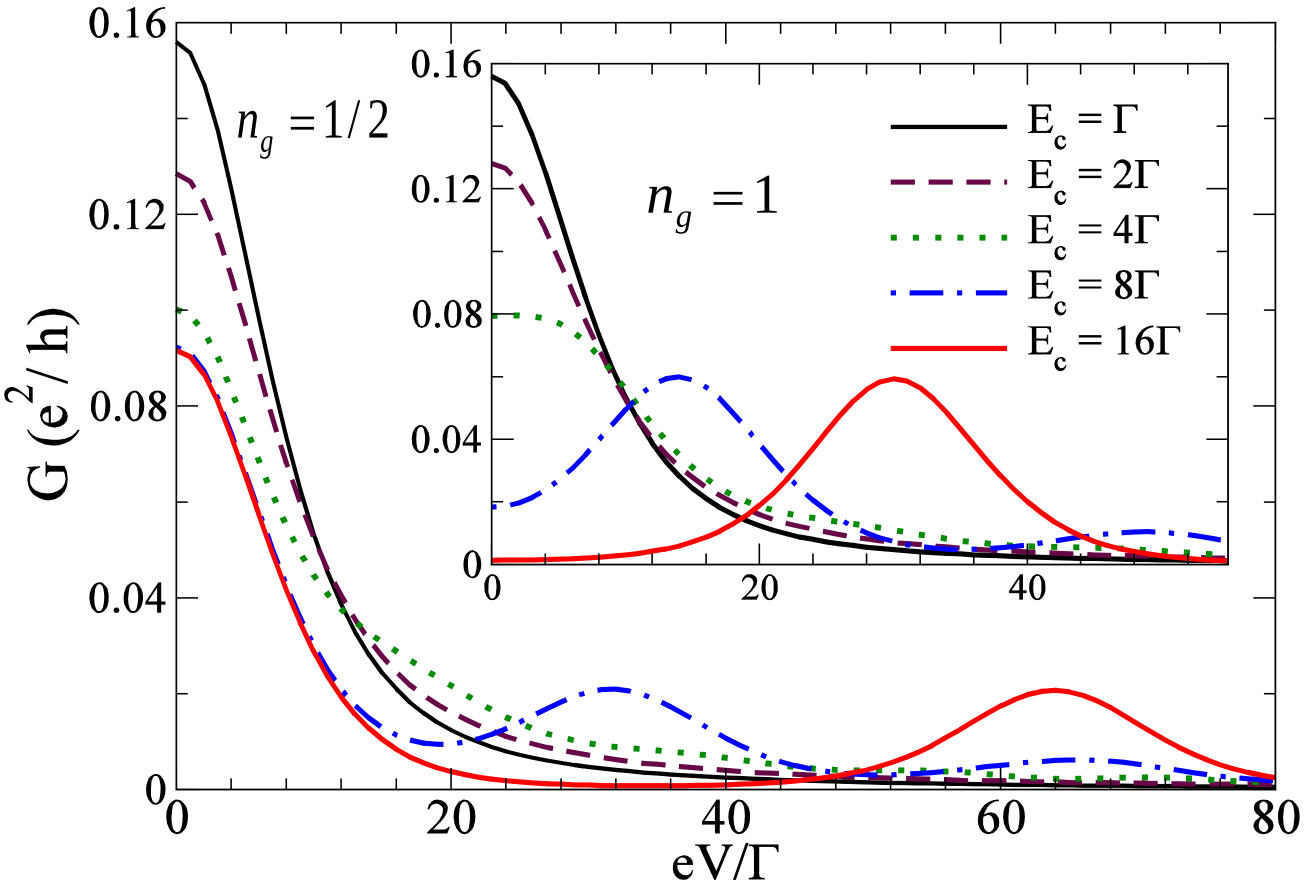
- Noninteracting peak value twice larger

- Strong thermal suppression of peak
- In addition interaction-induced suppression
  - Halved peak conductance in strong charging limit also for finite T

Peak conductance at T=0: from resonant Andreev reflection to teleportation



Finite bias sidepeaks



Master equation

$T = 2\Gamma$

Finite bias sidepeaks

- On resonance: sidepeaks at $eV = 4nE_C$
  - $\mu_{L,R}$ resonant with two (almost) degenerate higher order charge states: additional sequential tunneling contributions
  - Requires change of Cooper pair number – only possible due to MBSs:
    without Majoranas no side-peaks
- Similar sidepeaks away from resonance
- Peak location depends in characteristic way on magnetic field

Summary Part III

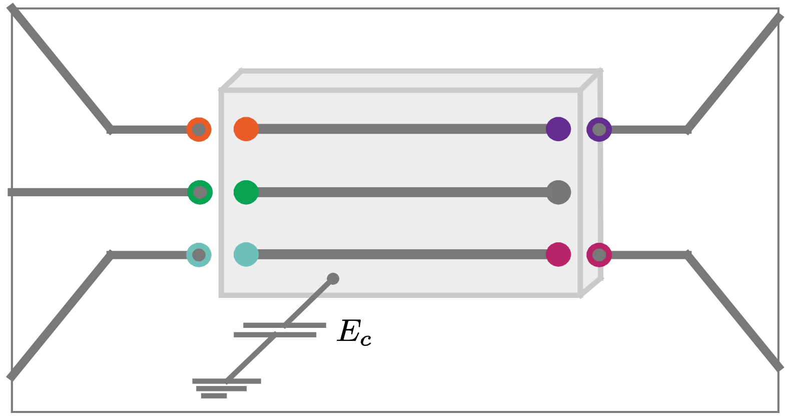
- Coulomb charging effects couple Cooper pair dynamics to Majorana fermions
- Simplest case: Majorana single-charge transistor (two MBSs)
- Teleportation vs resonant Andreev reflection
  - Nonlocality determines transport for strong charging energy
  - Crossover between teleportation and resonant Andreev reflection

Part IV: Topological Kondo effect

- For more than two MBSs on a floating SC: „quantum impurity spin“ nonlocally encoded by MBSs
- Couple „spin“ to normal leads: Cotunneling causes „exchange coupling“
- Stable non-Fermi liquid (multi-channel type) Kondo effect
- observable in electric conductance measurements

Beri & Cooper, PRL 2012
Altland & Egger, PRL 2013; Beri, PRL 2013
Altland, Beri, Egger & Tsvelik, PRL 2014
Zazunov, Altland & Egger, New J. Phys. 2014
Eriksson, Mora, Zazunov & Egger, PRL 2014
Buccheri, Babuijan, Korepin, Sodano & Trombettoni, Nucl. Phys. B 2015

Quantum impurity „spin“ with MBSs



- Now N>1 helical wires: M Majorana states tunnel-coupled to helical Luttinger liquid wires with g≤1
- Strong charging energy, with nearly integer n_g: unique equilibrium charge state on the island
- 2^{N-1}-fold ground state degeneracy due to Majorana states (taking into account parity constraint)
  - Need N>1 for interesting effect!

Parity constraint

- Uniqueness of equilibrium charge state implies parity constraint

$$Q = 2N_c + \sum_{\alpha=1}^{N} f_\alpha^+ f_\alpha = \mathrm{cst}$$

$$f_\alpha = \left(\gamma_{2\alpha-1} + i\gamma_{2\alpha}\right)/2$$

$$f_\alpha^+ f_\alpha \to 0{,}1$$

  - Degeneracy of Majorana sector is 2^N
  - Parity constraint $i^N \prod_{j=1}^{2N} \gamma_j = \pm 1$ removes half the states

- For now neglect MBS overlaps $\sim i\gamma_j\gamma_k$

Leads: Dirac fermion description

1D (spinless) helical liquid description of leads (j=1...M)

- Pair of right/left movers for x>0, with boundary condition $\psi_{j,L}(0)=\psi_{j,R}(0)$
- Low-energy Hamiltonian $v_F = 1$

$$H_{leads} = -i\sum_{j=1}^{M}\int_0^{\infty} dx \left(\psi^+_{j,R}\partial_x\psi_{j,R} - \psi^+_{j,L}\partial_x\psi_{j,L}\right) + \text{ee terms}$$

- Unfolding $\psi_L(x)=\psi_R(-x)$

$$H_{leads} = -i\sum_{j=1}^{M}\int_{-\infty}^{\infty} dx\ \psi^+_j\partial_x\psi_j + \text{ee terms}$$

Abelian bosonization

Convenient description of topological Kondo effect (even without interactions in the leads)

Electron operator is represented by dual pair of boson fields

$$\left[\phi_j(x), \theta_{j'}(x')\right]_- = -i\frac{\pi}{2}\delta_{jj'}\,\mathrm{sgn}(x - x')$$

Boson commutator ensures anticommutators in given lead

$$\psi_{j,R/L}(x) \sim \eta_j e^{i\left[\phi_j(x) \pm \theta_j(x)\right]}$$

$$\eta_j\eta_k + \eta_k\eta_j = 2\delta_{jk}$$

$$\eta_j\gamma_k + \gamma_k\eta_j = 0$$

Klein factors needed for anticommutators between different leads, represented by η Majorana fermions

Lead Hamiltonian

Bosonization gives Gaussian theory

$$H_{leads} = \sum_j \frac{1}{2\pi g} \int_0^\infty dx \, \left(g \left(\partial_x \phi_j \right)^2 + g^{-1} \left(\partial_x \theta_j \right)^2 \right)$$

e-e interactions in leads included „for free" through interaction parameter $1/2 < g \leq 1$ (weakly repulsive case): spinless Luttinger liquid

Noninteracting leads: $g = 1$

Nota bene:

Dirichlet boundary conditions at x=0 for „charge" fields *θ*
Neumann conditions for „phase" fields *Φ*

Klein-Majorana fusion

After gauge transformation: $H_T = \sum_{j=1..M} t_j \psi_j^+(0)\, e^{-i\varphi/2} \gamma_j + h.c.$

$$f_\alpha \to e^{-i\varphi/2} f_\alpha$$

$$\Longrightarrow \quad H_T = \sum_j t_j \left(i\eta_j\gamma_j\right)\, \sin\left(\phi_j(0) + \varphi/2\right)$$

Fuse Klein-Majorana and ‚true' Majorana at each contact

$$d_j = \left(\eta_j + i\gamma_j\right)/2 \qquad d_j^+ d_j = \left(1 + i\eta_j\gamma_j\right)/2 = 0{,}1$$

$\to$ all d fermion occupation numbers are conserved

(in absence of direct MBS couplings $\sim i\gamma_j\gamma_k$)

& can be gauged away

Dramatic simplification compared to standard „Luttinger liquid Y junction": **purely bosonic problem!**

Integrating out the leads

Euclidean functional integral: integrate out all boson fields away from x=0

$$\Phi_j = \phi_j(x=0)$$

$$Z = \sum_{W=-\infty}^{\infty} e^{2\pi i W n_g} \int D\varphi \; e^{-\frac{1}{E_C}\int d\tau \; \dot{\varphi}^2} \int D\Phi \; e^{-S}$$

$$S = \frac{Tg}{2\pi}\sum_{j,\omega}|\omega|\left|\Phi_j(\omega)\right|^2 + \sum_j t_j \int_0^{1/T} d\tau \;\; \sin\left(\Phi_j + \varphi/2\right)$$

$$\omega = 2\pi n T$$

Ohmic dissipation,
e-h pair excitations in leads

Tunneling from leads to
Majorana island

- Winding number $\varphi(\tau + 1/T) = 4\pi W + \varphi(\tau)$
- Near Coulomb valley: effectively only W=0 contributes

Phase action

- Shift boson fields $\Phi_j \to \Phi_j - \varphi/2$
  - Phase field φ is thereby gauged away in tunneling term
- Gaussian action for φ remains
  - Integration over φ can be done exactly...

$$S = \frac{Tg}{2\pi} \sum_{q=0}^{M-1} \sum_{\omega} \frac{|\omega|}{1 + \frac{2gME_C}{\pi|\omega|} \delta_{q,0}} \left| \tilde{\Phi}_q(\omega) \right|^2 + \sum_{j=1}^{M} t_j \int d\tau \sin \Phi_j(\tau)$$

$$\tilde{\Phi}_q = \frac{1}{\sqrt{M}} \sum_j e^{i2\pi qj/M} \Phi_j$$

Charging energy affects only isotropic phase field (q=0), which becomes „free“ at low energies

$$Z = \int D\Phi \; e^{-S}$$

Charging effects: dipole confinement

- High energy scales $> E_C$: charging effects irrelevant
  - Electron tunneling amplitudes renormalize independently upwards

$$t_j(E) \sim E^{-1+1/2g}$$

  - RG flow towards resonant Andreev reflection fixed point
- For $E < E_C$: charging induces ‚confinement'
  - In- and out-tunneling events are bound to ‚dipoles' with coupling $\lambda_{j\neq k}$: entanglement of different leads
  - Dipole coupling describes amplitude for cotunneling from lead j to lead k
  - ‚Bare' value

$$\lambda_{jk}^{(1)} = \frac{t_j(E_C)\, t_k(E_C)}{E_C} \sim E_C^{-3+1/g}$$

large for small E_C

QCD analogy

Phase field mode q=0 is „free“ at energies < E_C

- conjugate to pinned island charge, fluctuates strongly
- enforces finite lifetime $\sim E_C^{-1}$ of excited island states
  - In- and out-tunneling events separated by times of this order
  - Only virtual occupation of excited island states
- Particles (‚quarks‘) = in-tunneling events
- Antiparticles (‚antiquarks‘) = out-tunneling events

Particles and antiparticles bind together (dipoles or ‚mesons‘) at low energies: ‚confinement‘

but free at energies > E_C: ‚asymptotic freedom‘

RG equations in dipole phase

- Energy scales below E_C: effective phase action

$$S = \frac{g}{2\pi}\sum_j \int \frac{d\omega}{2\pi} |\omega| |\Phi_j(\omega)|^2 - \sum_{j\neq k} \lambda_{jk} \int d\tau \cos(\Phi_j - \Phi_k)$$

- One-loop RG equations

Lead DoS

$$\frac{d\lambda_{jk}}{dl} = -\left(g^{-1} - 1\right)\lambda_{jk} + \nu \sum_{m\neq(j,k)}^{M} \lambda_{jm}\lambda_{mk}$$

suppression by Luttinger tunneling DoS

enhancement by dipole fusion processes

- RG-unstable intermediate fixed point with isotropic couplings (for M>2)

$$\lambda_{j\neq k} = \lambda^* = \frac{g^{-1} - 1}{M - 2}\nu$$

Fixed points

Two stable fixed points: $\overline{\lambda} = 0, \infty$

Which one wins? Depends on $X = \lambda^{(1)} / \lambda^{*} \sim E_c^{-4+1/g}$

X<1: flow toward insulating junction $\overline{\lambda} = 0$

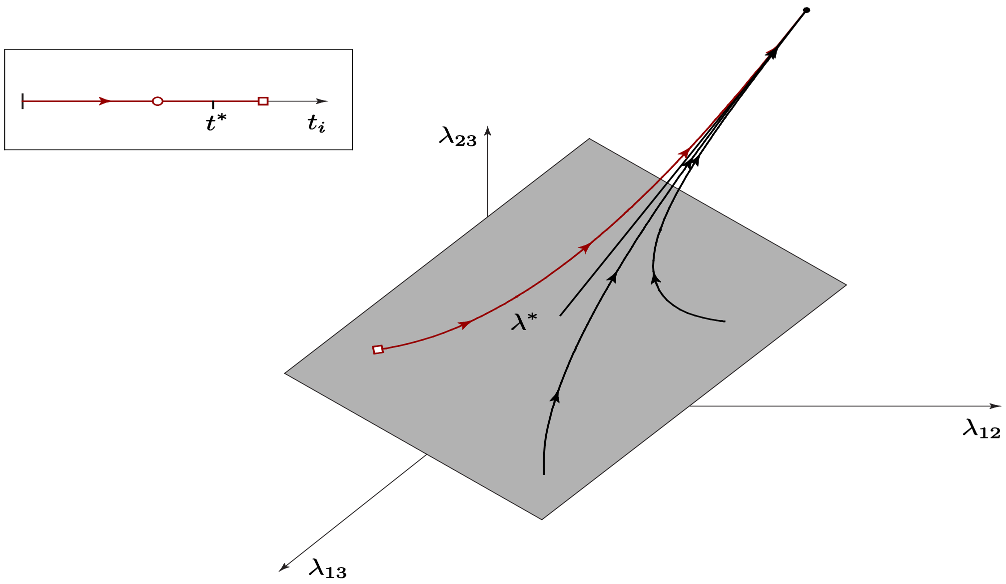
with vanishing conductance matrix $G_{jk} \sim T^{-2+2/g} \to 0$

X>1: isotropic flow to strong coupling $\overline{\lambda} = \infty$

exotic (non-Fermi liquid) Kondo regime

Resonant Andreev reflection fixed point is always unstable because of charging energy !

RG flow



- RG flow towards strong coupling for $\left\langle \lambda^{(1)} \right\rangle > \lambda^*$
  - Always happens for g=1 and/or moderate charging energy
- Flow towards isotropic couplings: anisotropies are RG irrelevant
  - implies stability of Kondo fixed point
- Perturbative RG fails below Kondo temperature

$$T_K \approx E_C e^{-\lambda^* / \left\langle \lambda^{(1)} \right\rangle}$$

Topological Kondo effect

Refermionize for g=1, use isotropic couplings

$$H = -i\int_{-\infty}^{\infty} dx \sum_{j=1}^{M} \psi_j^+ \partial_x \psi_j + i\lambda \sum_{j\neq k} \psi_j^+(0) S_{jk} \psi_k(0)$$

Majorana bilinears $S_{jk} = i\gamma_j \gamma_k$

- ‚Reality' condition: SO(M) symmetry [instead of SU(2)]
- nonlocal realization of ‚quantum impurity spin'
- Nonlocality ensures stability of Kondo fixed point

Majorana basis $\psi(x) = \mu(x) + i\xi(x)$ for leads:

$SO_2(M)$ Kondo model

$$H = -i\int dx \mu^T \partial_x \mu + i\lambda \mu^T(0) \hat{S} \mu(0) + [\mu \leftrightarrow \xi]$$

Example: Minimal case M=3

allows for spin-1/2 representation of „quantum impurity spin“

$$S_j = \frac{i}{4}\varepsilon_{jkl}\gamma_k\gamma_l$$

$$[S_1, S_2]_- = iS_3$$

- can be represented by standard Pauli matrices
- this spin is exchange coupled to effective spin-1 lead

$\rightarrow$ overscreened multi-channel Kondo effect

Expected: Residual ground state degeneracy, local non-Fermi liquid character

Towards strong coupling

On energy scales below Kondo temperature: phase fields are pinned near potential minima

$$S = \frac{g}{2\pi}\sum_j \int \frac{d\omega}{2\pi} |\omega| |\Phi_j(\omega)|^2 - \lambda \sum_{jk} \int d\tau \cos(\Phi_j - \Phi_k)$$

- Isotropic (q=0) phase field mode is decoupled, λ affects only M-1 orthogonal modes
- Low-energy physics governed by instantons connecting nearest-neighbor minima
- Flow from Neumann to Dirichlet conditions

Quantum Brownian Motion in periodic potential (hyper-triangular lattice) for particle with coordinate $\vec{\Phi}$

Dual boson theory

„Charge“ boson fields θ obey Neumann boundary conditions at strong coupling

- Need components „perpendicular“ to isotropic q=0 mode: constraint $\sum_j \Theta_j = 0$
- Gaussian fixed-point action plus leading irrelevant perturbation from instanton transitions

$$S = \frac{1}{2\pi g}\sum_j \int \frac{d\omega}{2\pi} |\omega| |\Theta_j(\omega)|^2 - w \sum_j \int d\tau \cos(2\Theta_j)$$

scaling dimension $y = 2g\frac{M-1}{M}$

always irrelevant (y>1) for g>1/2

Transport properties near unitary limit

- Temperature and voltage < T_K:

  Nonequilibrium Keldysh version of dual boson theory (include source fields)

- Linear conductance tensor

$$G_{jk} = e\frac{\partial I_j}{\partial \mu_k} = \frac{2e^2}{h}\left(1-\left(T/T_K\right)^{2y-2}\right)\left[\delta_{jk} - \frac{1}{M}\right]$$

  - Non-integer scaling dimension $y = 2g\left(1-\frac{1}{M}\right) > 1$

    implies non-Fermi liquid behavior even for g=1
  - completely isotropic multi-terminal junction

Correlated Andreev reflection

- Diagonal conductance at T=0 exceeds resonant tunneling („teleportation“) value but stays below resonant Andreev reflection limit

$$G_{jj} = \frac{2e^2}{h}\left(1 - \frac{1}{M}\right) \quad \Rightarrow \quad \frac{e^2}{h} < G_{jj} < \frac{2e^2}{h}$$

- Interpretation: Correlated Andreev reflection
- Remove one lead: change of scaling dimensions and conductance
- Non-Fermi liquid power-law corrections at finite T

Fano factor

Zazunov et al., NJP 2014

- Backscattering correction to current near unitary limit for $\sum_j \mu_j = 0$

$$\delta I_j = -\frac{e}{\hbar}\sum_k \left|\frac{\mu_k}{T_K}\right|^{2y-2}\left(\delta_{jk} - \frac{1}{M}\right)\mu_k$$

- Shot noise: $\tilde{S}_{jk}(\omega \to 0) = \int dt\, e^{i\omega t}\left(\langle I_j(t) I_k(0)\rangle - \langle I_j\rangle\langle I_k\rangle\right)$

$$\tilde{S}_{jk} = -\frac{2ge^2}{\hbar}\sum_l \left(\delta_{jl} - \frac{1}{M}\right)\left(\delta_{kl} - \frac{1}{M}\right)\left|\frac{\mu_l}{T_K}\right|^{2y-2} |\mu_l|$$

- universal Fano factor, but different value than for SU(N) Kondo effect

Sela et al. PRL 2006; Mora et al., PRB 2009

Summary Part IV

Coulomb-Majorana device with more than 2 MBSs allow for

„Topological Kondo effect“

with stable non-Fermi liquid behavior

Beri & Cooper, PRL 2012

Altland & Egger, PRL 2013

Zazunov, Altland & Egger, New J. Phys. 2014

Buccheri, Babujian, Korepin, Sodano & Trombettoni, Nucl. Phys. B 2015

Part V: Recent developments

- Probing the dynamics of the strongly entangled overscreened strong-coupling Kondo „impurity spin“ *Altland, Beri, Egger & Tsvelik, PRL 2014*
- Coupling the island in addition to another (grounded) superconductor:
  manifold of non-Fermi liquid states

  Eriksson, Mora, Zazunov & Egger, PRL 2014
- Networks of interacting Majorana fermions:
  Majorana surface code

  Xu & Fu, PRB 2010; Terhal, Hassler & Di Vincenzo, PRL 2012;
  Vijay, Hsieh & Fu, arXiv:1504.01724; Plugge et al. (in preparation)

Majorana spin dynamics

Altland, Beri, Egger & Tsvelik, PRL 2014

- Overscreened multi-channel Kondo fixed point: massively entangled effective impurity degree remains at strong coupling: „Majorana spin"
- Probe and manipulate by coupling of MBSs

$$H_Z = \sum_{jk} h_{jk} S_{jk}$$

  - ‚Zeeman fields' $h_{jk} = -h_{kj}$ describe overlap of MBS wavefunctions within same nanowire
  - Zeeman fields couple to $S_{jk} = i\gamma_j\gamma_k$

Majorana spin near strong coupling

Bosonized form of Majorana spin at Kondo fixed point:

$$S_{jk} = i\gamma_j\gamma_k \cos\left[\Theta_j(0) - \Theta_k(0)\right]$$

- Dual boson fields $\Theta_j(x)$ describe ‚charge' (not ‚phase') in respective lead
- Scaling dimension $y_Z = 1 - \frac{2}{M}$ $\rightarrow$ RG relevant
- Zeeman field ultimately destroys Kondo fixed point & breaks emergent time reversal symmetry
- Perturbative treatment possible for $T_h < T < T_K$

dominant 1-2 Zeeman coupling: $T_h = \left(\frac{h_{12}}{T_K}\right)^{M/2} T_K$

Crossover SO(M)→SO(M-2)

- Lowering T below T_h $\longrightarrow$ crossover to another Kondo model with SO(M-2) (Fermi liquid for M<5)
  - Zeeman coupling h_{12} flows to strong coupling $\longrightarrow$ γ_1, γ_2 disappear from low-energy sector
  - Same scenario follows from Bethe ansatz solution

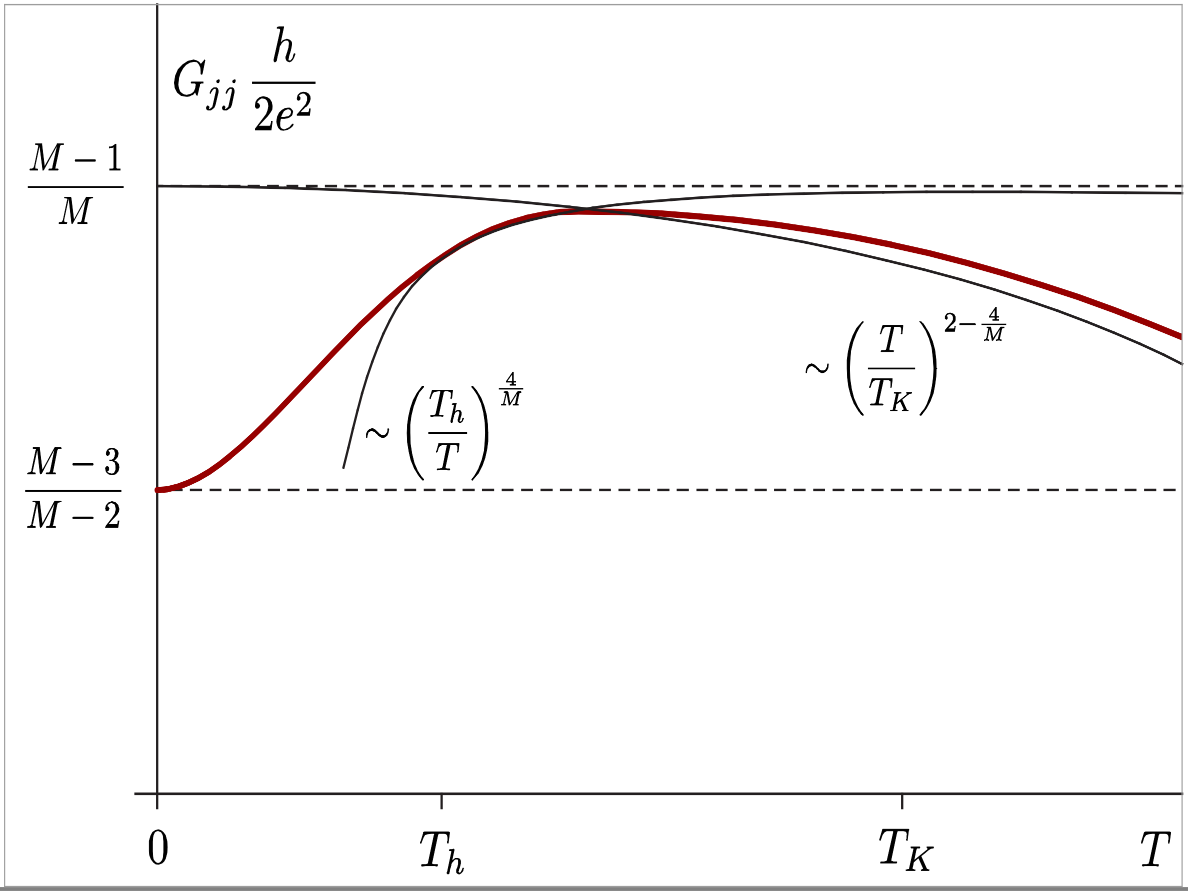
Altland, Beri, Egger & Tsvelik, JPA 2014

- Observable in conductance & in thermodynamic properties

SO(M)→SO(M-2): conductance scaling

for single Zeeman component $h_{12} \neq 0$ consider G_{jj} $(j \neq 1,2)$

(diagonal element of conductance tensor)



Multi-point correlations

- Majorana spin has nontrivial multi-point correlations at Kondo fixed point, e.g. for M=3 (absent for SU(N) case)

$$\left\langle T_\tau S_j(\tau_1) S_k(\tau_2) S_l(\tau_3) \right\rangle \sim \frac{\varepsilon_{jkl}}{T_K (\tau_{12}\tau_{13}\tau_{23})^{1/3}}$$

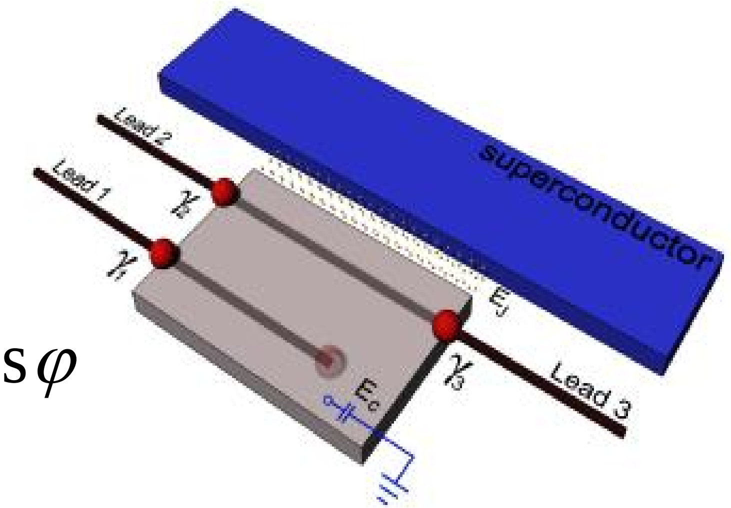
- Observable consequences for time-dependent ‚Zeeman' field $B_j = \varepsilon_{jkl} h_{kl}$ with $\vec{B}(t) = \left(B_1 \cos(\omega_1 t), B_2 \cos(\omega_2 t), 0\right)$
  - Time-dependent gate voltage modulation of tunnel couplings
  - Measurement of ‚magnetization' by known read-out methods
  - Nonlinear frequency mixing $\left\langle S_3(t) \right\rangle \sim B_1 B_2 \cos\left[(\omega_1 \pm \omega_2) t\right]$
  - Oscillatory transverse spin correlations (for B_2=0)

$$\left\langle S_2(t) S_3(0) \right\rangle \sim B_1 \frac{\cos(\omega_1 t)}{(\omega_1 t)^{2/3}}$$

Adding Josephson coupling: Non Fermi liquid manifold

Eriksson, Mora, Zazunov & Egger, PRL 2014

$$H_{island} = E_C\left(2N_c + \hat{n} - n_g\right)^2 - E_J \cos\varphi$$



with another bulk superconductor: Topological Cooper pair box

Effectively harmonic oscillator for $E_J >> E_C$

with Josephson plasma oscillation frequency $\Omega = \sqrt{8E_J E_C}$

Low energy theory

- Tracing over phase fluctuations gives two coupling mechanisms:
  - Resonant Andreev reflection processes

$$H_A = \sum_j t_j \gamma_j \left(\psi_j^+(0) - \psi_j(0) \right)$$

  - Kondo exchange coupling, but of $SO_1(M)$ type

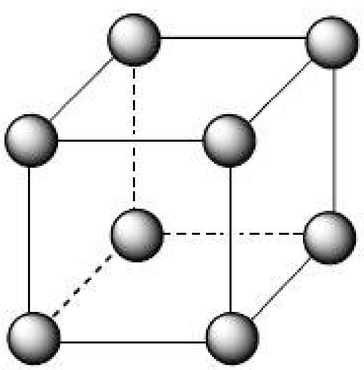
$$H_K = \sum_{j \neq k} \lambda_{jk} \left(\psi_j^+(0) + \psi_j(0) \right)\left(\psi_k^+(0) + \psi_k(0) \right) \gamma_j \gamma_k$$

- Interplay of resonant Andreev reflection and Kondo screening for $\Gamma < T_K$

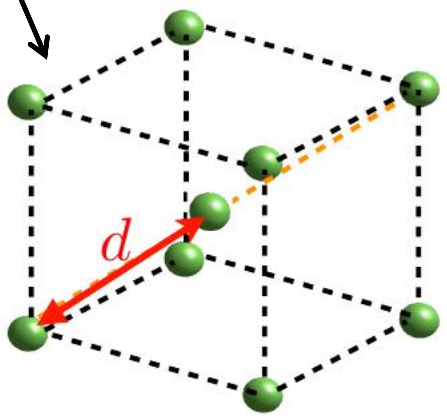
Quantum Brownian Motion picture

Abelian bosonization now yields (M=3)

$$H_A + H_K \propto -\sum_j \sqrt{\Gamma_j} \sin \Phi_j - \sqrt{T_K} \sum_{j \neq k} \cos \Phi_j \cos \Phi_k$$



Simple cubic lattice



bcc lattice

Quantum Brownian motion

- Leading irrelevant operator (LIO): tunneling transitions connecting nearest neighbors
- Scaling dimension of LIO from n.n. distance d

$$y_{LIO} = \frac{d^2}{2\pi^2}$$

Yi & Kane, PRB 1998

  - Pinned phase field configurations correspond to Kondo fixed point, but unitarily rotated by resonant Andreev reflection corrections
- Stable non-Fermi liquid manifold as long as LIO stays irrelevant, i.e. for $y_{LIO} > 1$

Scaling dimension of LIO

- M-dimensional manifold of non-Fermi liquid states spanned by parameters $\delta_j = \sqrt{\Gamma_j / T_K}$
- Scaling dimension of LIO

$$y = \min\left\{2, \frac{1}{2}\sum_{j=1}^{M}\left[1 - \frac{2}{\pi}\arcsin\left(\frac{\delta_j}{2(M-1)}\right)\right]\right\}$$

  - Stable manifold corresponds to y>1
  - For y<1: standard resonant Andreev reflection scenario applies
  - For y>1: non-Fermi liquid power laws appear in temperature dependence of conductance tensor

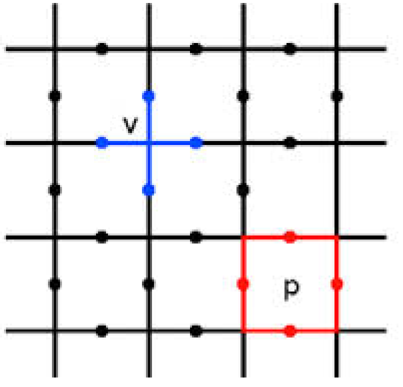
Majorana surface code

- Recent interest on networks of interacting Majorana fermions
  - perform topological (and universal) quantum computation ?
- Surface code architecture
  - Encode logical qubit through many physical qubits, with topological protection
  - Error detection via classical „software“
  - Superconducting qubits: cumbersome and complicated, but at present most promising approach *Fowler, Mariantoni, Martinis & Clarke, PRA* ***86****, 032324 (2012)*

Paradigm: Kitaev toric code

Kitaev & Laumann, arXiv:0904.2771

2D toric code: exactly solvable spin-1/2 model on square lattice



$$H = -J_e \sum_v A_v - J_m \sum_p B_p$$

$$A_v = \prod_{j \in \mathrm{star}(v)} \sigma_j^x \qquad B_p = \prod_{j \in \partial p} \sigma_j^z$$

- All star and plaquette operators commute and have eigenvalues ±1
- Ground state: $A_v |\Psi\rangle = B_p |\Psi\rangle = |\Psi\rangle$

Intrinsic topological order

- On surface of genus g: ground state has degeneracy 4^g
- Quasiparticle spectrum:
  - „electric“ charges (flip star operator) and „magnetic“ vortices (flip plaquette operator)
  - individually behave as bosons
  - But: nontrivial mutual statistics
  - Abelian anyons

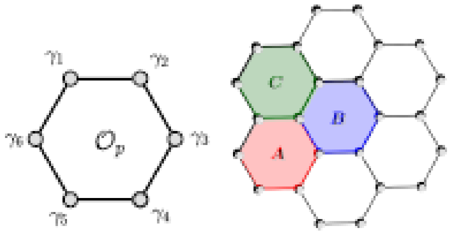
Majorana surface code

Xu & Fu, PRB 2010; Terhal et al. PRL 2012; Vijay, Hsieh & Fu, arXiv:1504.01724

Majorana plaquette model

$$H = -u \sum_p O_p$$

$$O_p = i \prod_{j \in \partial p} \gamma_j$$



e.g. for honeycomb lattice, but other lattices also work

All plaquette operators mutually commute, eigenvalues ±1

Ground state is gapped and follows from $O_p \left|\Psi\right\rangle = \left|\Psi\right\rangle$

Z_2 intrinsic topological order

- On torus (periodic boundary conditions in both directions): Fourfold GS degeneracy
- Indicates intrinsic topological order
- Proof: Count degrees of freedom and constraints
  - $2^{N/2-1}$ d.o.f.: for N MBS, we have $2^{N/2}$ dim Hilbert space with conserved total parity Γ
  - Constraints: $$\prod_{p\in A} O_p = \prod_{p\in B} O_p = \prod_{p\in C} O_p = \Gamma \equiv i^{N/2} \prod_j \gamma_j$$

    Each plaquette type (ABC) causes $2^{N/6-1}$ GS constraints

$$\Rightarrow \quad D = \frac{2^{N/2-1}}{\left(2^{N/6-1}\right)^3} = 4$$

Anyon excitations

- Elementary plaquette excitations (A,B,C) plus composite objects (AB, BC, AC, ABC)
- Elementary excitation (A,B,C) have bosonic self-statistics, but Berry phase π under exchange of different types
- ABC equals the corner-shared Majorana operator
- Plaquettes can be flipped only in pairs!

How to realize the Majorana plaquette model experimentally ?

- Our proposal:

Plugge, Landau, Sela, Albrecht, Altland & Egger, in preparation

Use Coulomb-Majorana islands as in topological Kondo effect, but form network of such islands connected by tunneling contacts

- Lowest-order excitations yield plaquette Hamiltonian & realize Kitaev toric code
- Read-out and manipulation of plaquettes by simple conductance measurements

see talk by S. Plugge @ Natal workshop

Summary Part V

- Probing the dynamics of strongly entangled overscreened strong-coupling Kondo „impurity spin“ *Altland, Beri, Egger & Tsvelik, PRL 2014*
- Coupling the island to another (grounded) superconductor: Manifold of non-Fermi liquid states *Eriksson, Mora, Zazunov & Egger, PRL 2014*
- Networks of interacting Majorana fermions: Majorana surface code

Summary of this course:

1. Majorana fermions and Majorana bound states (MBSs): Basics
2. Kitaev chain: Basics and realization
3. Majorana takes charge
   Coupling Cooper pairs and Majorana fermions through Coulomb charging effects
4. Topological Kondo effect
   Stable overscreened multi-channel Kondo effect
5. Recent developments

THANK YOU FOR YOUR ATTENTION!